REVISED
SECOND EDITION

CALCULUS I

Guided Notebook

JOHN R. TAYLOR ▪ DESIRÉ J. TAYLOR

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Review of Algebra and Trigonometry

Section 1.2

A. Functions and Relations

DEFINITION **Relation**: A set of ordered pairs.
$(x, y) \rightarrow$ (domain, range)

DEFINITION **Function**: A correspondence from one set (the domain) to another set (the range) such that each element in the domain corresponds to exactly one element in the range.

Example: Determine whether each of the following is an example of a function or not.

1. $\{(1,1)(3,2)(5,3)\}$

2. $\{(1,1)(2,4)(3,-5)(2,-4)\}$

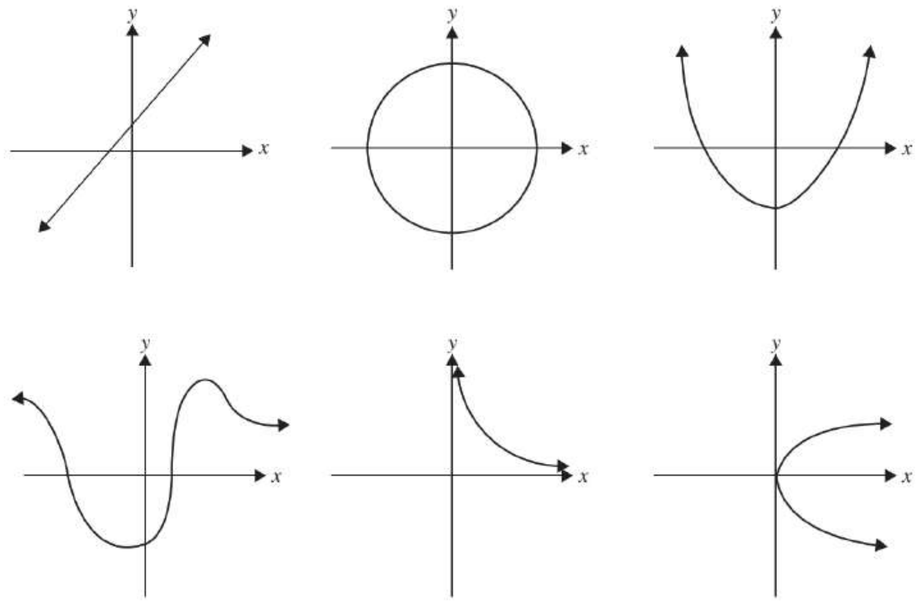
3. $x^2+2y=0$

4. $x^2+y^2=1$

Vertical Line Test for Functions

Vertical Line Test for Functions: If any vertical line intersects a graph more than once, then the graph is **not** a function.

Example: Determine whether each of the following is a function or not by the Vertical Line Test.



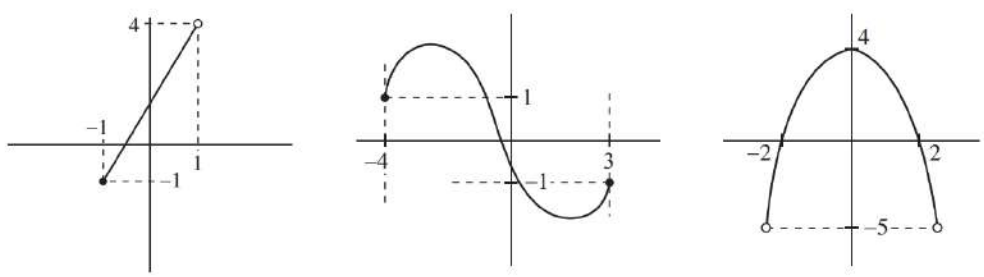
B. Domain and Range of a Function

DEFINITION **Domain**: Input $\rightarrow x$ values (i.e., all of the values of x that I may plug into a function)

DEFINITION **Range**: Output $\rightarrow y$ values (i.e., all of the values of y that a function can attain)

Example: Give the domain and range (in interval notation) for each of the following:

1.



Domain:__________ Domain:__________ Domain:__________

Range: __________ Range: __________ Range: __________

2. $g(x)=\sqrt{2-x}$

3. $f(x)=\dfrac{x^2-4}{x+2}$

4. $g(x)=\dfrac{3x}{\sqrt{x+5}}$

5. $h(x)=\ln(6x-3)$

*The three functions for which we will most frequently have domain restrictions (in this course) are **fractions**, **radicals,** and **logarithms**.

C. Linear Models

Definition

$$\textbf{Slope}=m=\frac{rise}{run}=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$$

Slope	$m=\dfrac{y_2-y_1}{x_2-x_1}$	(Use when given two points to find slope)
Slope–Intercept Form	$y=mx+b$	(Use when given slope and y intercept)
Point–Slope Form	$(y-y_0)=m\cdot(x-x_0)$	(Use when given one point and slope)
General Form	$A\cdot x+B\cdot y+C=0$	
Horizontal Line	$y=b$ (where $b=$ constant)	
Vertical Line	$x=c$ (where $c=$ constant)	

Parallel Lines

Two lines are parallel if and only if they have the same slope.

For two lines $y_1=m_1x+b_1$ and $y_2=m_2x+b_2$ we have $y_1 \parallel y_2 \iff m_1=m_2$

Perpendicular lines

Two lines are perpendicular if and only if the product of their slope $= -1$.

For two lines $y_1 = m_1 x + b_1$ and $y_2 = m_2 x + b_2$ we have $y_1 \perp y_2 \quad \Leftrightarrow \quad m_1 = -\dfrac{1}{m_2}$

Examples: Find the equation of the line:

1. That passes through point $(0, -3)$ with slope $= -2$

2. That passes through points $(3, -2)$ and $(4, 5)$

3. That passes through point $(0, 0)$ and is parallel to the line $y = 3x - 9$

4. That passes through point $(2, -4)$ and is perpendicular to the line $y = \dfrac{1}{2}x + 3$

5. Find the slope and y intercept of the line $9x - 3y - 3 = 0$

D. Classes of Functions

1. Power Functions

For any real number m, a function in the form $f(x) = x^m$ is called a Power Function

2. Polynomials

Definition

A *polynomial function* is a function in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ where $a_0 \neq 0$and n is a positive integer.

Examples: State whether each is a polynomial:

1. $g(x) = x^2 + 5x + 6$

2. $f(x) = x^3 + 7x - \sqrt{x}$

3. $h(x) = x^{\frac{2}{3}} - x + 5$

4. $f(x) = \dfrac{x^2}{3} + 8x$

3. Rational Functions

A Rational Function is the quotient of two polynomial functions:

A Rational Function is a function of the form $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0}$

Asymptotes

An asymptote is an imaginary line that the graph of a function approaches as the function approaches a restricted number in the domain or as it approaches infinity.

I. Locating Vertical Asymptotes

If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, $p(x)$and $q(x)$ *have no common factors* and n is a zero of $q(x)$, then the line $x = n$ is a vertical asymptote of the graph of $f(x)$.

II. Locating Horizontal Asymptotes

Let $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x \ + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x \ + b_0}$

i. If $n < m$, then $y = 0$ is the horizontal asymptote.

ii. If $n = m$, then the line $y=\dfrac{a_n}{b_m}$ is the horizontal asymptote.

iii. If $n > m$, there is NO horizontal asymptote. (But there will be a slant/oblique asymptote.)

Examples: Find all asymptotes.

1. $f(x)=\dfrac{15x}{3x^2+1}$

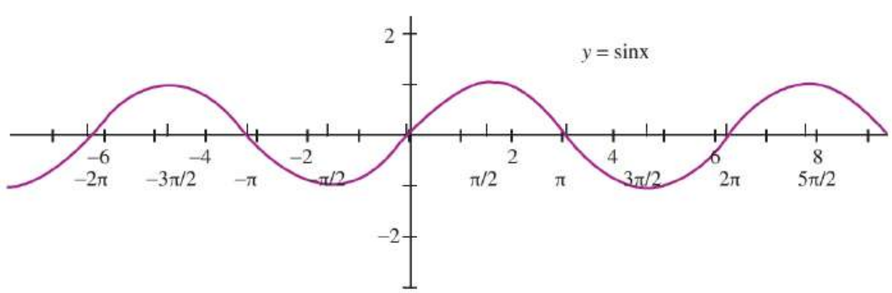
2. $g(x)=\dfrac{15x^3}{3x^2+1}$

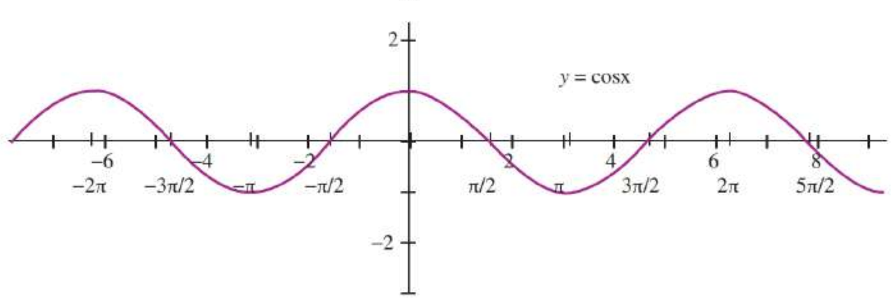
3. $h(x)=\dfrac{-3x+7}{5x-2}$

4. $k(x)=\dfrac{2x-x^2}{x^2-2x-3}$

4. Trigonometric Functions

sin x	csc x
cos x	sec x
tan x	cot x





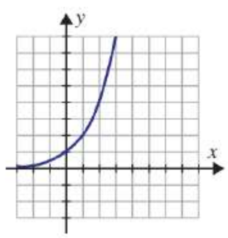
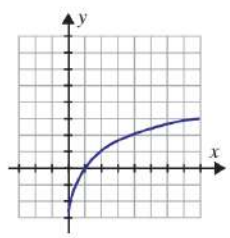
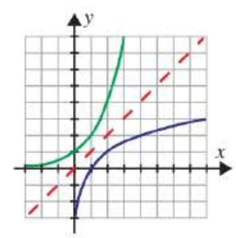
5. Exponential and Logarithmic Functions

DEFINITION: An exponential function is a function in the form $f(x) = a^x$ (i.e., the variable x is in the exponent).

DEFINITION: A logarithmic function is a function in the form $f(x) = \log_a x$ (i.e., the variable x is in the expression).

$\boldsymbol{y = \log_b x}$ "***y*** is equal to log base ***b*** of ***x***"—Here "***b***" is the BASE NUMBER and "***x***" is the VARIABLE.

$\log_b x = y$ means exactly the same thing as $b^y = x$

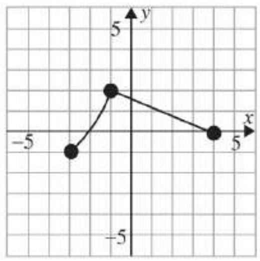
$y = 2^x$	$y = \log_2(x)$	**Comparison of the two graphs, showing the inversion line in red.**
		

E. Transformations of Functions

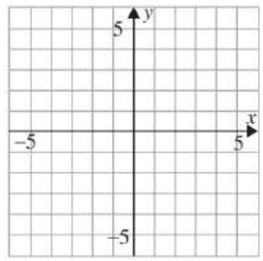
Vertical Shifts	$f(x)+C$ $f(x)-C$	↑ ↓	Moves graph **UP C** units Moves graph **DOWN C** units
Horizontal Shifts	$f(x-C)$ $f(x+C)$	→ ←	Moves graph **RIGHT C** units Moves graph **LEFT C** units
Vertical and Horizontal Reflections	$-f(x)$ $f(-x)$	↕ ↔	Flips graph about ***x*-axis** Flips graph about ***y*-axis**
Vertical Stretching/ Compressing	$c \cdot f(x)$ for $c > 1$ $c \cdot f(x)$ for $0 < c < 1$	↑↓ ↓↑	Graph vertically **stretches** by a factor of **C** Graph vertically **shrinks** by a factor of **C**

Example: Use the given graph of $f(x)$ to sketch each of the following:

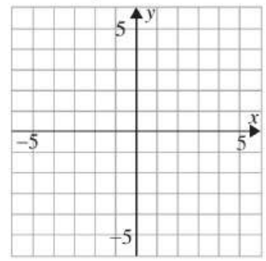
$f(x)$



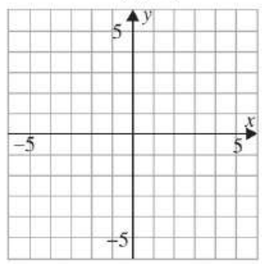
a. $f(x)+2$



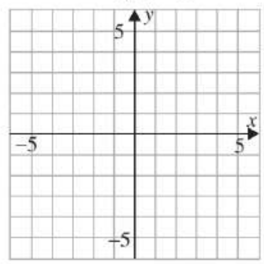
b. $f(x+2)$



c. $-f(x)$



d. $2f(-x)$

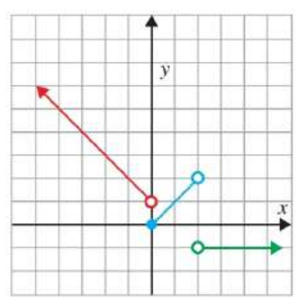


F. Combinations of Functions

1. *Piecewise-Defined Functions*

A Piecewise Function is a function that has specific (and different) definitions on specific intervals of x.

$$f(x)=\begin{cases} -x+1 & x<0 \\ x & 0\le x<2 \\ -1 & x>2 \end{cases}$$



2. *Sums, Differences, Products, and Quotients of Functions*

Sum	$(f+g)(x)=f(x)+g(x)$
Difference	$(f-g)(x)=f(x)-g(x)$
Product	$(f\cdot g)(x)=f(x)\cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

3. *Composition of Functions*

$f(x)=x^2+5x+2$	$f(☺)=(☺)^2+5(☺)+2$
$g(x)=\frac{2x}{\sqrt{x+1}}$	$g(\quad)=\frac{2(\quad)}{\sqrt{(\quad)+1}}$
$k(x)=2x-3$	$k(\quad)=2(\quad)-3$
$h(x)=\sqrt{x^2+5x}$	$h(\qquad)=\sqrt{(\qquad)^2+5(\qquad)}$ $=\sqrt{(\qquad)^2+5(\qquad)}$

Notation	$(f\circ g)(x)=f(g(x))$

Example

1. For the functions $f(x)=\sqrt{x}$ and $g(x)=x+2$ find

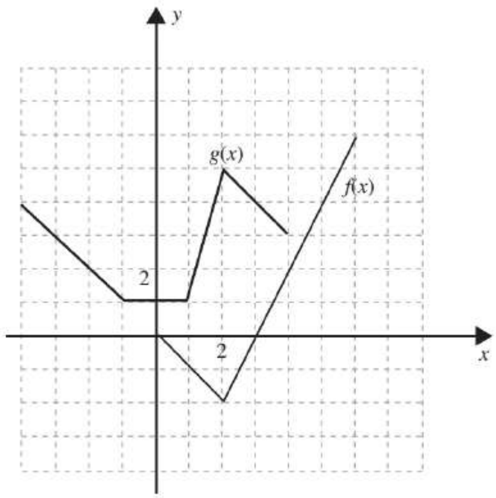
a. $(f\circ g)(x)=$

b. $(g\circ f)(x)=$

c. $(f\circ g)(2)=$

d. $(g\circ g)(x)=$

Example: For the functions $f(x)$ and $g(x)$ given in the graph find



a. $(f \circ g)(2) =$

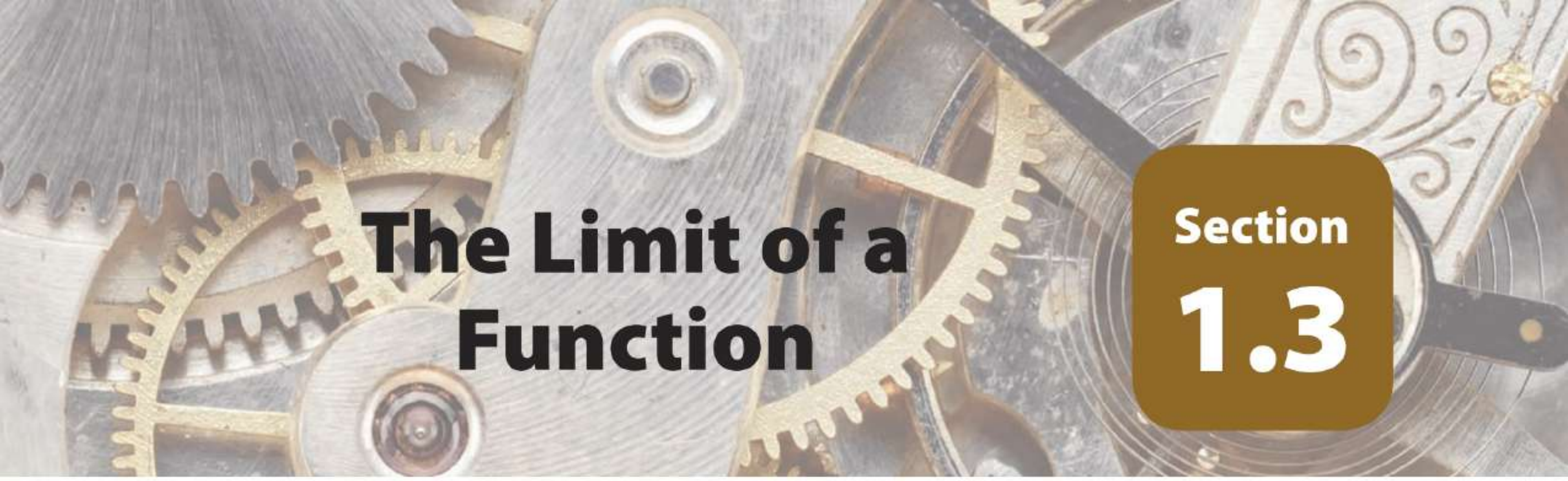
b. $(f \circ g)(3) =$

c. $(g \circ f)(2) =$

d. $(g \circ f)(3) =$

e. $(f \circ f)(4) =$

f. $(g \circ g)(4) =$



The Limit of a Function

Section 1.3

A. Limits

DEFINITION: $\lim_{x \to a} f(x) = L$ The limit of $f(x)$ as x approaches a, equals L. (Where is the functions value headed as x is "on its way" to a?)

$\lim_{x \to a^-} f(x)$ The limit of $f(x)$ as x approaches a from the LEFT.

$\lim_{x \to a^+} f(x)$ The limit of $f(x)$ as x approaches a from the RIGHT.

B. Techniques of Solving Limits

1. Evaluation

When possible (without violating domain rules) "plug it in."

Example

1. **(Video)** $\lim_{x \to 3} x^2 + 2x - 3 =$

2. **(Video)** $\lim_{x \to 1} 5 - \frac{1}{x} =$

3. $\lim_{x \to 3} x^2 =$

4. $\lim_{x \to 1} \frac{1}{x} =$

2. Factoring/Manipulation (then Evaluation)

Factor expressions and cancel any common terms.

Example

5. **(Video)** $\lim_{x \to 4} \frac{3x - 12}{x^2 - 16} =$

6. **(Video)** $\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 5x + 6} =$

7. $\lim_{x \to 4} \frac{x-4}{x^2-16} =$

8. $\lim_{x \to 3} \frac{x^2-3x}{x^2-2x-3} =$

3. Table

Set up a table as x approaches the limit from the left and from the right.

Example

9. **(Video)** $\lim_{x \to 0} \frac{1-\cos x}{x} =$

10. $\lim_{x \to 0} \frac{\sin x}{x} =$

$x \to 0^-$	-0.1	-0.01	-0.001
$\frac{\sin x}{x}$			

$x \to 0^+$	0.1	0.01	0.001
$\frac{\sin x}{x}$			

IDENTITY: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

4. Graphing

Graph the function and inspect. (Warning: Your graphing calculator might not always indicate a hole or small discontinuity in a graph. Be sure to always check the domain for restrictions.)

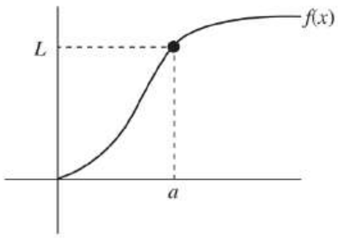
Example

11. $\lim_{x \to 0} \frac{\sin x}{x} =$

12. $\lim_{x \to 0^+} \frac{1}{x} =$

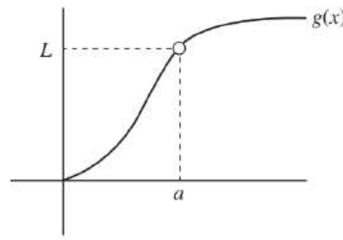
More Examples

13.



$f(a) =$

$\lim\limits_{x \to a} f(x) =$



$g(a) =$

$\lim\limits_{x \to a} g(x) =$

14.

$$f(x) = \begin{cases} 7 - x & \text{if } x \leq -4 \\ x & \text{if } -4 < x \leq 2 \\ (x-1)^2 & \text{if } x > 2 \end{cases}$$

$\lim\limits_{x \to -4^-} f(x)$

$\lim\limits_{x \to -4^+} f(x)$

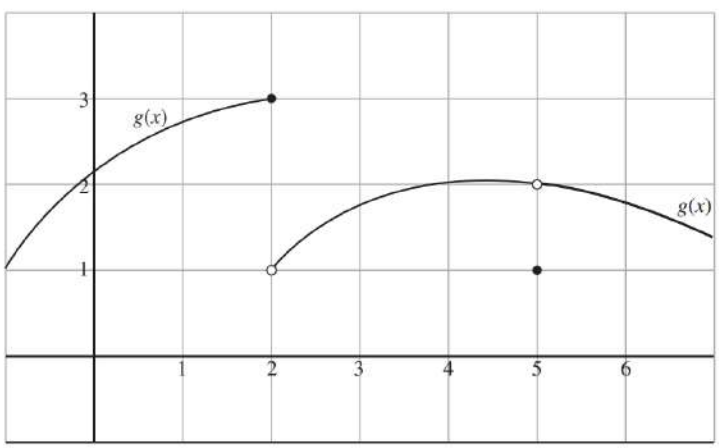
$\lim\limits_{x \to -4} f(x)$

$\lim\limits_{x \to 2^-} f(x)$

$\lim\limits_{x \to 2^+} f(x)$

$\lim\limits_{x \to 2} f(x)$

15.



$g(2) =$

$\lim\limits_{x \to 2^-} g(x) =$

$\lim\limits_{x \to 2^+} g(x) =$

$\lim\limits_{x \to 2} g(x) =$

$g(5) =$

$\lim\limits_{x \to 5^-} g(x) =$

$\lim\limits_{x \to 5^+} g(x) =$

$\lim\limits_{x \to 5} g(x) =$

C. Average Velocity

DEFINITION: $Velocity = \dfrac{Distance}{Time}$

Example

16. A ball is thrown straight up into the air at an initial velocity of 75 ft/sec, its height in feet t seconds is given by $y = 75t - 16t^2$.

a. Find the average velocity for the period beginning when $t = 2$ and lasting

i. 0.1 seconds (i.e., the time period [2, 2.1])

ii. 0.01 seconds

iii. 0.001 seconds

b. Estimate the instantaneous velocity of the ball when $t = 2$.

Calculating Limits

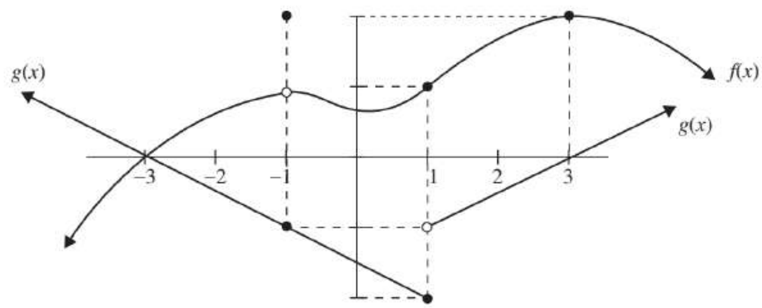
Section 1.4

A. Limit Laws

1. Assume that f and g are functions and c is a constant.
2. $\lim\limits_{x\to a}\left(f(x)+g(x)\right)=\lim\limits_{x\to a} f(x)+\lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a}\left(f(x)-g(x)\right)=\lim\limits_{x\to a} f(x)-\lim\limits_{x\to a} g(x)$
4. $\lim\limits_{x\to a}\left(c\cdot f(x)\right)=c\cdot\left(\lim\limits_{x\to a} f(x)\right)$
5. $\lim\limits_{x\to a}\left(f(x)\cdot g(x)\right)=\lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$
6. $\lim\limits_{x\to a}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$; $\lim\limits_{x\to a} g(x)\neq 0$
7. $\lim\limits_{x\to a}\left(f(x)\right)^n=\left(\lim\limits_{x\to a} f(x)\right)^n$
8. $\lim\limits_{x\to a} c=c$
9. $\lim\limits_{x\to a} x=a$
10. $\lim\limits_{x\to a} x^n=a^n$
11. $\lim\limits_{x\to a} \sqrt[n]{x}=\sqrt[n]{a}$
12. $\lim\limits_{x\to a} \sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a} f(x)}$

Example

1. **(video)**



a. $\lim_{x\to 1^+} 3g(x) =$

b. $\lim_{x\to 1} f(x) =$

c. $\lim_{x\to 1} \left(f(x)+g(x)\right) =$

d. $\lim_{x\to 3} \left(f(x)\cdot g(x)\right) =$

2. **(video)** Given $\lim_{x\to a} h(x) = 1$, $\lim_{x\to a} f(x) = 10$ and $\lim_{x\to a} g(x) = 0$

a. $\lim_{x\to a} \frac{h(x)}{f(x)}$

b. $\lim_{x\to a} f(x)^{-1}$

c. $\lim_{x\to a} \sqrt{f(x)}$

d. $\lim_{x\to a} \frac{1}{f(x)-g(x)}$

e. $\lim_{x\to a} \frac{g(x)}{h(x)}$

B. Calculating Limits

Direct Substitution Property: If f is a polynomial or rational function and $a \in$ Domain, then

$$\lim_{x \to a} f(x) = f(a)$$

Examples

3. $\lim\limits_{x \to 5} \dfrac{2x+2}{x-3} =$

4. $\lim\limits_{\theta \to \pi} \cos\theta =$

Indeterminate Form: If $\lim\limits_{x \to a} f(x) = f(a) = \dfrac{0}{0}$ then ***factor***, ***simplify***, or multiply by the ***conjugate***

5. $\lim\limits_{x \to 4} \dfrac{x^2-16}{x-4} =$

6. $\lim\limits_{x \to 0} \dfrac{(x+2)^2-4}{2x} =$

7. $\lim\limits_{x \to 0} \dfrac{\sqrt{x+16}-4}{x} =$

Undefined Form: If $\lim\limits_{x \to a} f(x) = f(a) = \dfrac{Number}{0}$ then the limit does not exist—***DNE***.
You can support your answer by graphing or using the table method to show.

8. $\lim_{x \to 5} \frac{x^2 - x - 6}{x - 5} =$

Theorem—We say that a limit exists when the limit from the left equals the limit from the right.

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Examples

9. **(video)** $h(x) = \begin{cases} x^2 + 3x - 1 & \text{if } x \neq 2 \\ x - 1 & \text{if } x = 2 \end{cases}$

Find $\lim_{x \to 2} h(x)$

10. $h(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ -1 & \text{if } x = 2 \end{cases}$

Find $\lim_{x \to 2^-} h(x) =$

$\lim_{x \to 2^+} h(x) =$

$\lim_{x \to 2} h(x) =$

Theorem—Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ when x is near a, then $\lim_{x \to a} g(x) = L$

Examples

11. Find $\lim\limits_{x \to 0} x \cdot \cos\left(\frac{1}{x}\right)$

More Examples

12. $\lim\limits_{x \to -8^-} \frac{5x+40}{|x+8|} =$

$\lim\limits_{x \to -8^+} \frac{5x+40}{|x+8|} =$

$\lim\limits_{x \to -8} \frac{5x+40}{|x+8|} =$

13. Evaluate $\lim_{x \to 0} \dfrac{\tan(5x)}{\sin(6x)} =$

Continuity

Section **1.5**

A. Definition of Continuity

DEFINITION: A function f is continuous at a number a if

i. $f(a)$ exists

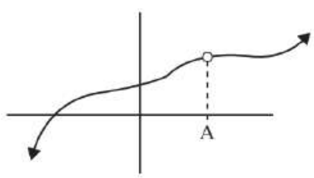
ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

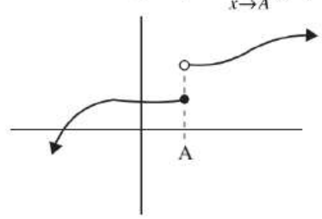
A function is defined as continuous only if it is continuous at every point in the domain of the function.

Examples: For each, determine whether the function is continuous (i.e., is $\lim_{x \to A} f(x) = f(A)$?)

1.



2.



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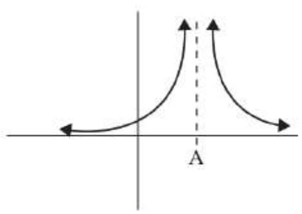

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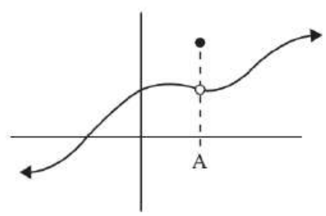


Examples: For each, determine whether the function is continuous. If not, where is the discontinuity?

5. **(video)** $f(x)=\dfrac{x^2-x-20}{x-5}$

6. **(video)** $f(x)=\begin{cases} -6-x & \text{if } x\le -3 \\ x & \text{if } -3<x\le 3 \\ (x-1)^2 & \text{if } x>3 \end{cases}$

7. **(video)** $h(x)=\begin{cases} x^2+1 \ \ if\ x\ne 2 \\ 5\ \ if\ x=2 \end{cases}$

Examples: For each, determine the value of c so that function is continuous for all values of x.

8. **(video)** $f(x)=\begin{cases} cx^2+3x-7 & x\in(-\infty,3] \\ 2x+1 & x\in(3,\infty) \end{cases}$

9. $f(x)=\begin{cases} cx+7 & x\in(-\infty,8) \\ cx^2-7 & x\in[8,\infty) \end{cases}$

Where are these functions discontinuous?

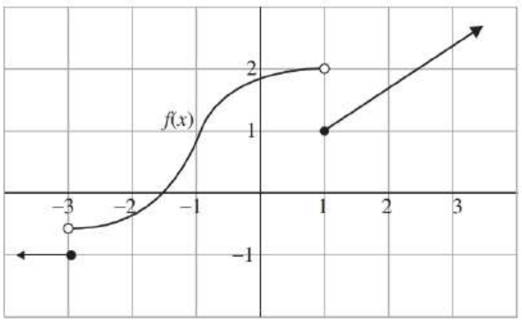
10. $f(x)=\dfrac{\ln x+\sin x}{x^2-1}$

11. $f(x)=\ln(1+\cos x)$

DEFINITION: A function f is continuous from the RIGHT at a number a if $\lim_{x \to a^+} f(x) = f(a)$
A function f is continuous from the LEFT at a number a if $\lim_{x \to a^-} f(x) = f(a)$

Example

12.



f is continuous from the LEFT or RIGHT at

a. $x = -3$

b. $x = 1$

13. Show that $f(x)$ has a jump discontinuity at $x = 9$ by calculating the limits from the left and right at $x = 9$.

$$f(x) = \begin{cases} x^2 + 5x + 5 & \text{if } x < 9 \\ 14 & \text{if } x = 9 \\ -4x + 4 & \text{if } x > 9 \end{cases}$$

Theorem—If f and g are functions that are continuous at a number a, and c is a constant, then the following are also continuous at a:

i. $(f + g)$

ii. $(f - g)$

iii. $(f \cdot g)$

iv. $\left(\frac{f}{g}\right)$ if $g(a) \neq 0$

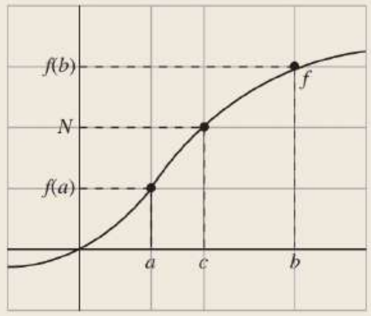
v. $c \cdot f$ or $c \cdot g$

Theorem—A **polynomial** function is continuous everywhere

A **rational** function is continuous everywhere it is defined

Theorem—Intermediate Value Theorem

If f is a function that is continuous on a closed interval $[a,b]$ where $f(a) \neq f(b)$ and N is a number such that $f(a) < N < f(b)$. Then there exist a number c such that $a < c < b$ and $f(c) = N$.



Examples

14. Show that $f(x) = x^2 - x - 2$ has a root on the interval $[1,3]$

15. Let f be a continuous function such that $f(1) < 0 < f(9)$. Then the Intermediate Value Theorem implies that $f(x) = 0$ on the interval (A, B). Give the values of A and B.

Limits Involving Infinity

Section 1.6

A. Infinity versus DNE

Recall from Section 1.3 that $\lim_{x\to 0} \frac{1}{x^2}$ DNE since the function value kept increasing. Now we will be more descriptive; any value that keeps increasing is said to approach infinity (∞), and any value that keeps decreasing is said to approach negative infinity ($-\infty$).

Examples

1. **(video)** Evaluate $\lim_{x\to 0} \frac{1}{x^2}$ using the graph and table method.

$\lim_{x\to 0^-} \frac{1}{x^2}$	
x	y
-0.1	
-0.01	
-0.001	

$\lim_{x\to 0^+} \frac{1}{x^2}$	
x	y
0.1	
0.01	
0.001	

2. **(video)** Evaluate $\lim_{x\to 0} \frac{1}{x}$ using the graph and table method.

$\lim_{x\to 0^-} \frac{1}{x}$	
x	y
-0.1	
-0.01	
-0.001	

$\lim_{x\to 0^+} \frac{1}{x}$	
x	y
0.1	
0.01	
0.001	

B. A Quick Review of Asymptotes

An asymptote is an imaginary line that the graph of a function approaches as the independent variable approaches a restricted number or infinity.

Locating Vertical Asymptotes

If $f(x)=\dfrac{p(x)}{q(x)}$ is a rational function, $p(x)$ and $q(x)$ have no common factors and k is a zero of $q(x)$, then the line $x=k$ is a vertical asymptote of the graph of $f(x)$.

Locating Horizontal Asymptotes

$$\text{Let } f(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

i. If $n<m$, then $y=0$ is the horizontal asymptote.

ii. If $n=m$, then the line $y=\dfrac{a_n}{b_m}$ is the horizontal asymptote.

iii. If $n>m$, there is NO horizontal asymptote. (But there may be a slant/oblique asymptote.)

Examples: For the following rational functions, find the vertical and horizontal asymptotes if any:

3. $f(x)=\dfrac{16x^2}{4x^2+1}$

4. $g(x)=\dfrac{x+8}{x^2-64}$

5. $h(x)=\dfrac{x^3+7}{5x-2}$

6. $k(x)=\dfrac{x^2-2x}{2-3x+x^2}$

C. Vertical Asymptotes

Vertical asymptotes occur when $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$
The asymptote will be the line $x = a$.

Example: Evaluate the limit, find the asymptote and graph the function

7. **(video)** $\lim_{x\to 2} \frac{2x+1}{(3x-6)^2}$

8. **(video)** $\lim_{x\to -4^-} \frac{x+6}{x+4}$ $\quad$ $\lim_{x\to -4^+} \frac{x+6}{x+4}$ $\quad$ $\lim_{x\to -4} \frac{x+6}{x+4}$

9. $\lim_{x\to 2} \frac{x+1}{2x-4}$

D. Limits to Infinity

A limit as the domain approaches infinity: $\lim_{x \to \infty} f(x)$

Finding Limits to Infinity of Rational Functions

i. Determine the degree of the denominator. (Let's say degree = P)

ii. Multiply both the numerator and denominator by $\frac{1}{x^P}$.

iii. Distribute/clean up algebra and continue evaluating the limit.

Example: Evaluate the limit.

10. **(video)** $\lim_{x \to \infty} \frac{6x^2 + 2x + 7}{8x + 2x^2}$

11. **(video)** $\lim_{x \to \infty} \frac{x^3 + 4x - 2}{6 - 2x^2}$

12. **(video)** $\lim_{x \to \infty} \frac{2x}{2x^2 + x - 1}$

Conclusion: For positive integers M *and* N *such that* $M > N$

1. Degree of the Numerator = Degree of the Denominator

$$\lim_{x \to \infty} \frac{\textit{Polynomail of Degree } M}{\textit{Polynomail of Degree } M} = \textit{Ratio of Leading Coeficients}$$

2. Degree of the Numerator > Degree of the Denominator

$$\lim_{x \to \infty} \frac{\textit{Polynomail of Degree } M}{\textit{Polynomail of Degree } N} = \pm\infty$$

3. Degree of the Numerator < Degree of the Denominator

$$\lim_{x \to \infty} \frac{\textit{Polynomail of Degree } N}{\textit{Polynomail of Degree } M} = 0$$

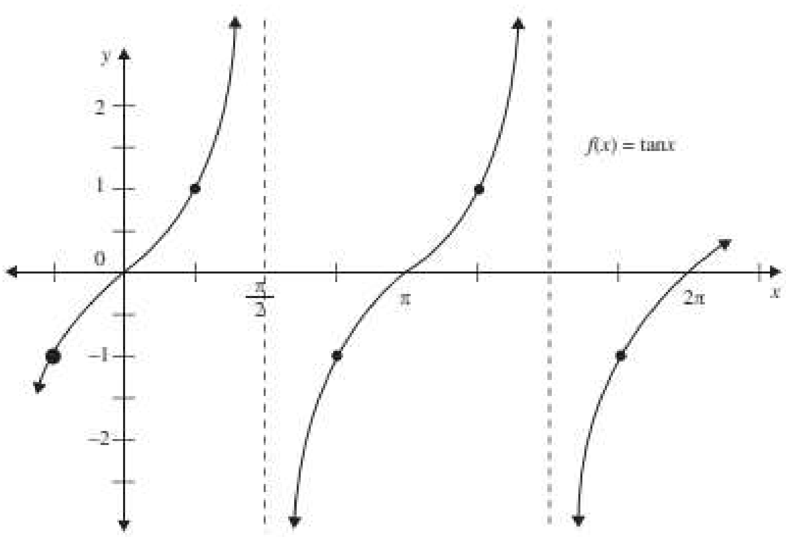
More Examples: Evaluate the limit.

13. $\lim\limits_{x\to\infty} \dfrac{3x-10}{\sqrt{16x^2+5}}$

14. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{9x^2+x+1}}{2x+1}$

15. $\lim\limits_{x\to\frac{\pi}{2}^-} \tan x$

16. $\lim\limits_{x\to\frac{\pi}{2}^+} \tan x$



17. $\lim_{x \to \infty} \sqrt{x^2 + 7x + 1} - x$

Derivatives and Rates of Change

Section **2.1**

A. Slope of Secant Functions

Recall: Slope $= m = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$. From this we are able to derive:

Slope of the Secant Line to a Function: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

1. **(video)** **a.** Find the **slope** of the secant line to the function $f(x) = x^2 + 3x - 5$ between $x = 1$ and $x = 2$.

b. **(video)** Find the **equation** of the secant line to the function $f(x) = x^2 + 3x - 5$ between $x = 1$ and $x = 2$.

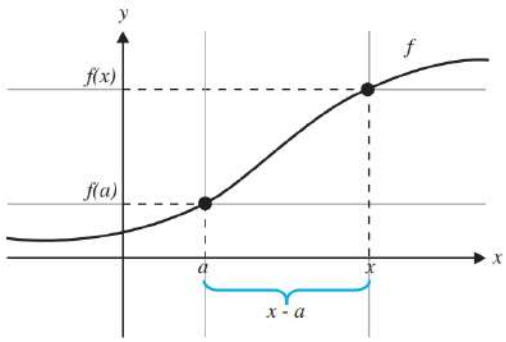
2. **a.** Find the **slope** of the secant line to the function $f(x) = \sqrt{x}$ between $x = 1$ and $x = 9$.

b. Find the **equation** of the secant line to the function $f(x)=\sqrt{x}$ between $x=1$ and $x=9$.

3. Estimate the slope of the **tangent** line to the function $y=x^2$ at the point $(1,1)$ by calculating the slope of the **secant** line between $x=1$ and $x=1.1$, between $x=1$ and $x=1.01$, and between $x=1$ and $x=1.001$.

A generalization of the previous example gives the definition: Slope $= \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$. Similarly we can approach this by defining the distance between x and a as $x - a = h$.

Slope of the tangent line of $f(x)$ at point specific a



In order to get the two points P_1 and P_2 as close together as possible, we need the space $x - a \to 0$.

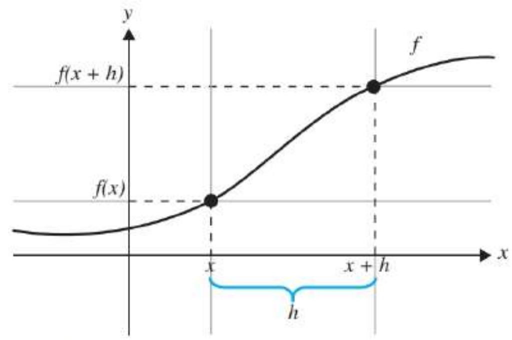
So, the slope between P_1 and P_2 is:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

but as the space $x - a \to 0$, we have

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Slope of the tangent line of $f(x)$ at a general point x



In order to get the two points P_1 and P_2 as close together as possible, we need the space $h \to 0$.

So, the slope between P_1 and P_2 is:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

$$= \frac{f(x+h) - f(x)}{h}$$

but as the space $h \to 0$, we have

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

B. Definition of Derivative

The definition of a derivative (aka the slope of the tangent function) is given as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{or} \qquad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*Note: In this section, we will use the DEFINITION OF THE DERIVATE to calculate all derivatives. (This means we will be doing it the *long way*!)

Examples

4. **(video)** Find the equation of the tangent line to the function $f(x) = x^2 + 3x + 10$ where $x = 5$.

5. **(video)** Find the derivative of the function $f(x) = x^2 + 3x + 10$ using the difference quotient and the definition of derivative.

6. Find the equation of the tangent line to the function $f(x) = x^2 - 3x + 1$ where $x = 5$.

7. A person standing on top of a 200 ft. tall building throws a ball into the air with a velocity of 96 ft/sec. The function $s(t) = -16t^2 + 96t + 200$ gives the ball's height above ground, t seconds after it was thrown.Find the instantaneous velocity of the ball at $t = 2$ seconds

8. The position of a particle is given by the values of the tables

t(seconds)	0	1	2	3	4	5
s(feet)	0	14	47	51	86	103

Find the average velocity for the time period beginning when $t = 2$ and lasting

1. 3 s (i.e., for the time interval [2,5])

2. 2 s

3. 1 s

9. a. The equation of the tangent line to the graph of $y = g(x)$ at $x = 3$ if $g(3) = -3$ and $g'(3) = 7$ is $y =$

b. If the tangent line to $y = f(x)$ at $(2,10)$ passes through the point $(0,4)$,

then $f(2) =$

and $f'(2) =$

10. $\lim_{h\to 0}\frac{\sqrt{81+h}-9}{h}$ represents the derivative of the function $f(x)=$______at the number $a=$______

11. $\lim_{x\to 6}\frac{2^x-64}{x-6}$ represents the derivative of the function $f(x)=$______ at the number $a=$ ______

The Derivative of a Function

Section

2.2

A. Definition of the Derivative

For a function $f(x)$ the derivative is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Examples: Using the definition of the derivative, find the derivative of the following functions:

1. (video) $f(x) = 2x^2 + 8x - 2$

2. (video) $g(x) = 2\sqrt{x} + 9$

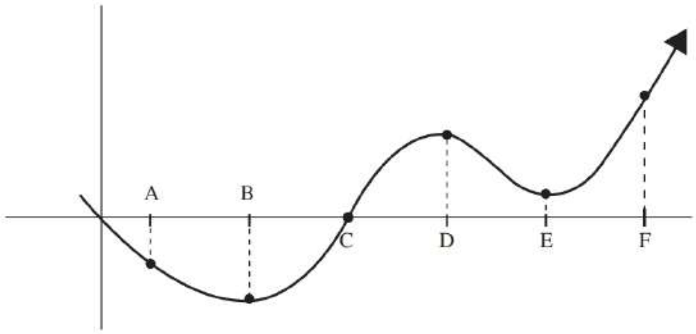
3. **(video)** $h(x)=\dfrac{5}{3x+1}$

4. $f(x)=x^2+3x-2$

5. $k(x)=\dfrac{2}{x}$

6. $g(x)=\sqrt{2x+1}$

7. Consider the graph for the function $f(x)$



Estimate the following:

a. $f'(A) =$

b. $f'(B) =$

c. $f'(C) =$

d. $f'(D) =$

e. $f'(E) =$

f. $f'(F) =$

B. Notation

Function	Derivative
$y =$	$y' =$ $\frac{dy}{dx} =$ (Leibniz notation)
$f(x)$	$f'(x)$
$F(x)$	$f(x)$

C. Differentiability

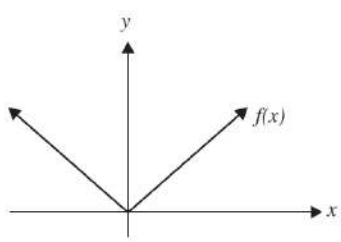
DEFINITION: We say that

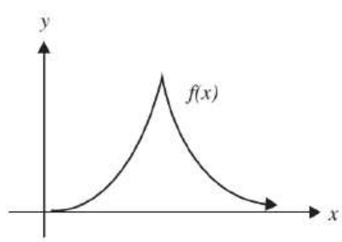
- $f(x)$ is differentiable at a if $f'(a)$ exists
- $f(x)$ is differentiable on (a, b) if it is differentiable on every point in (a, b)

Theorem: If f is differentiable at a, then f is continuous at a

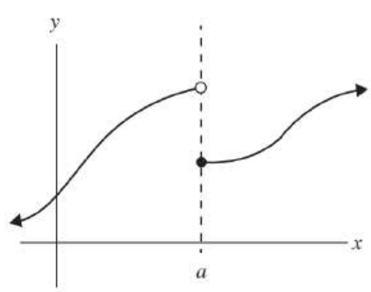
Video Nondifferentiable functions

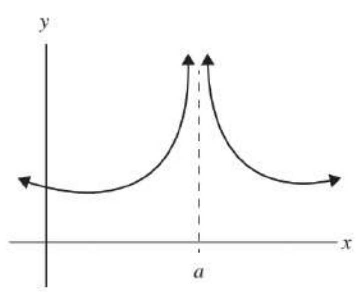
i. Any function with a "corner" or cusp



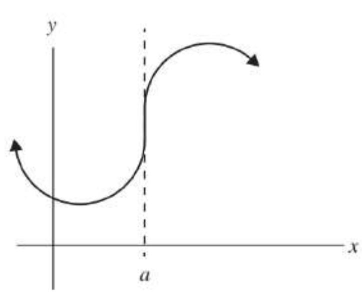


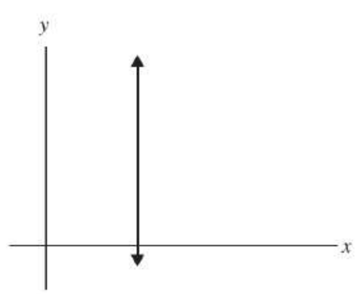
ii. Any function with a discontinuity





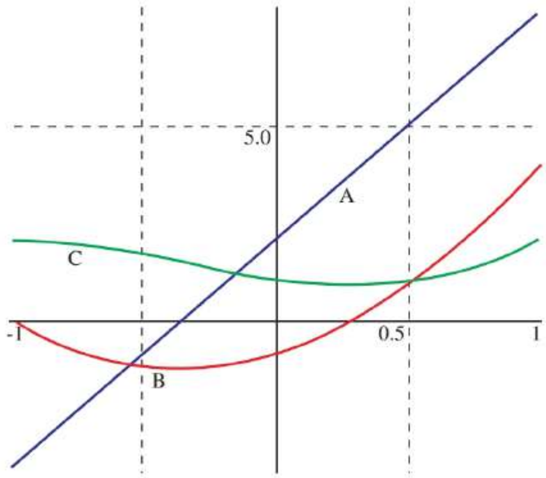
iii. Any function with a vertical tangent





More Examples

8.



Identify the graphs A (blue), B (red), and C (green) as the graphs of a function and its derivatives.

a. The graph of the function is:

b. The graph of the function's first derivative is:

c. The graph of the function's second derivative is:

9. Evaluate the limit $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^2-1}$

10. Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+36}-6}{x^2}$.

Basic Differentiation

Section
2.3

A. Properties and Formulas (The short way—Yeah!)

1. Basic Functions

Function	Derivative
$f(x)=c$ (Constant)	$f'(x)=0$
$f(x)=x$	$f'(x)=1$
$f(x)=cx$	$f'(x)=c$
$f(x)=x^n$	$f'(x)=nx^{n-1}$
$f(x)=cx^n$	$f'(x)=cnx^{n-1}$
$f(x)=c\cdot g(x)$	$f'(x)=c\cdot g'(x)$
$f(x)=g(x)+h(x)$	$f'(x)=g'(x)+h'(x)$
$f(x)=g(x)-h(x)$	$f'(x)=g'(x)-h'(x)$

Note: For the function $f(x)=g(x)\cdot h(x)$ we CANNOT say that $f'(x)=g'(x)\cdot h'(x)$

For the function $f(x)=\dfrac{g(x)}{h(x)}$ we CANNOT say that $f'(x)=\dfrac{g'(x)}{h'(x)}$

2. Trigonometric Functions

Function	Derivative
$f(x)=\sin x$	$f'(x)=\cos x$
$f(x)=\cos x$	$f'(x)=-\sin x$
$f(x)=\tan x$	$f'(x)=\sec^2 x$
$f(x)=\csc x$	$f'(x)=-\csc x\cot x$
$f(x)=\sec x$	$f'(x)=\sec x\tan x$
$f(x)=\cot x$	$f'(x)=-\csc^2 x$

Examples: Find and LABEL the derivatives of each of the following functions.

1. (video) $f(x)=17$

2. (video) $y=3x$

3. (video) $k(x)=x^5$

4. (video) $h(x)=5x^3$

5. (video) $s(r)=4\pi r^2$

6. (video) $\frac{d}{dx}\left(\sqrt{x}\right)$

7. (video) $\frac{d}{dx}\left(4x^{\frac{3}{2}}\right)$

8. (video) $v(t)=\frac{1}{t}$

Common Derivatives

$$\frac{d\left(\sqrt{x}\right)}{dx}=\frac{1}{2\sqrt{x}} \qquad \text{Since} \qquad \frac{d\left(\sqrt{x}\right)}{dx}=\frac{d\left(x^{\frac{1}{2}}\right)}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$$

$$\frac{d\left(\frac{1}{x}\right)}{dx}=-\frac{1}{x^2} \qquad \text{Since} \qquad \frac{d\left(\frac{1}{x}\right)}{dx}=\frac{d\left(x^{-1}\right)}{dx}=-x^{-2}=-\frac{1}{x^2}$$

9. **(video)** Find the derivative $d(x)=\frac{9}{x^3}+3x+2$

10. **(video)** Find the derivative $g(t)=5t^2+\frac{4}{t^2}-18$

11. **(video)** Find the derivative $g(t)=4t^3-\sqrt{t}+2$

12. $p(x)=\frac{x^3+4x^2}{x}$

13. $d(x)=(x+3)^2$

14. $f(x)=\sin x+\tan x$

15. $g(x)=2\cos x+\sec x$

16. $f(x)=\sqrt[3]{x}-\cot x$

More Examples

17. $f(x)=x^3+4x^2-2x+4$

Find $f'(x)=$

$f'(2)=$

Find $f''(x)=$

$f''(2)=$

18. $g(t)=\dfrac{4t^2+t+5}{\sqrt{t}}$

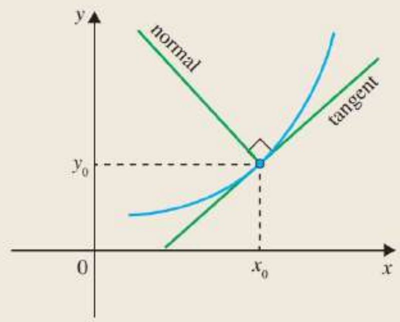
Find $g'(t)=$ $g'(1)=$

Find $g''(t)=$ $g''(1)=$

19. $h(x)=x^5-2x^4+3x^3-x-6$ Find the first six derivatives of the function.

B. Normal and Tangent Lines to a Function

A line that is perpendicular to the tangent line at the point of intersection



Examples

20. Find the horizontal tangent lines (lines with slope = 0) to the function $f(x) = 2x^3 + 3x^2 - 120x + 23$

21. Find the tangent and the normal lines to the function $f(x) = 4\cos x$ at $x = \dfrac{\pi}{3}$

C. Applications to Position, Velocity, and Acceleration

If the motion/position function of a particle is known, we can find the velocity and acceleration functions in the following way.

- If the **position** of a particle is given by $f(x)$, then the **velocity** of the particle is given by $f'(x)$
- If the **velocity** of a particle is given by $g(x)$, then the **acceleration** of the particle is given by $g'(x)$

  (We can also say that if the position of a particle is given by $f(x)$, then the acceleration of the particle is given by $f''(x)$, the second derivative of the motion function.)

Alternative notation:

- **Position** of a particle $s(t)$
- **Velocity** of a particle $v(t)$
- **Acceleration** of a particle $a(t)$

Then $v(t)=s'(t)$ and $a(t)=v'(t)=s''(t)$

Example

22. A particle's **position** is described by the function $s(t)=3t^3-144t$ (t is measures in seconds and $s(t)$ in feet.)

a. Find the velocity function.

b. Find the acceleration function.

c. Find the acceleration after 9 seconds.

d. Find the acceleration when the velocity is 0.

23. A particle's **position** is described by the function $s(t)=t^3-9t^2+15t+10$ (t is measures in seconds and $s(t)$ in feet.)

a. Find the velocity function.

b. What is the velocity after 3 seconds?

c. When is the particle at rest?

d. When is the particle moving in a positive direction?

e. When is the particle slowing down?

f. Find the total distance traveled during the first 8 seconds.

More Examples (Webwork)

24. The area of a disc with radius r is $A(r) = \pi r^2$. Find the rate of change of the area of the disc with respect to its radius when $r = 5$.

25. If $f(x) = \left(\frac{3}{4}x\right)^9$, then $f'(x) =$

26. If $f(x) = \sqrt{x} \cdot (3x + 2)$, then $f'(x) =$

27. **a.** If $f(x) = 12\pi^2$, then $f'(x) =$

b. $f(x) = 12x^2$, then $f'(x) =$

c. $f(x) = 12\pi x^2$, then $f'(x) =$

28. If a ball is thrown vertically upward from the roof of 64-foot building with a velocity of 32 ft/sec, its height after t seconds is $s(t) = 64 + 32t - 16t^2$

a. What is the maximum height the ball reaches?

b. What is the velocity of the ball when it hits the ground (height 0)?

Product and Quotient Rules

Section **2.4**

A. Product Rule

$$f(x)=g(x)\cdot h(x) \text{ then } f'(x)=g'(x)\cdot h(x)+g(x)\cdot h'(x)$$

Alternative Notation: $\dfrac{d(g\cdot h)}{dx}=\left(\dfrac{dg}{dx}\right)\cdot(h)+(g)\cdot\left(\dfrac{dh}{dx}\right)$

In Plain English: The derivative of the product of two functions (which we will call the "first" function and the "second" function) is equal to the **derivative of the first, times the second, plus the first, times derivative of the second**.

Examples

A. $f(x)=\left(2x^3+8x\right)\cdot\left(5x^4+17\right)$

We see that $f(x)$ consists of the product of two smaller functions, in this case $\left(2x^3+8x\right)$ "the first" and $\left(5x^4+17\right)$ "the second." So, the derivative then is:

$$f'(x)=\underbrace{\left(6x^2+8\right)}_{\textbf{Derivative of the first}}\times\underbrace{\left(5x^4+17\right)}_{\textbf{The second}}+\underbrace{\left(2x^3+8x\right)}_{\textbf{The first}}\times\underbrace{\left(20x^3\right)}_{\textbf{Derivative of the second}}$$

Note: You should leave the answer in this form unless we are asked to “clean up.”
Again, do not forget to label your derivative.

B. $g(x) = x \cdot \sin x$

We see that $g(x)$ consists of the product of two smaller functions; in this case, x “the first” and $\sin x$ “the second.” So, the derivative then is: $g'(x) = (1) \cdot \sin x + x \cdot \cos x = \sin x + x \cdot \cos x$

More Examples: Find and LABEL the derivatives of each of the following functions.

1. **(video)** $f(x) = (3x^5 + 2\sqrt{x}) \cdot (4x^2 + 51)$

2. **(video)** $f(x) = 7x^2 \sin x$

3. $f(x) = \left(x^4 + \sqrt{x}\right) \cdot (5x - 1)$

4. $f(x) = x^2 \cos x$

5. $f(x) = \sin x \cos x$

B. Quotient Rule

$$f(x) = \frac{g(x)}{h(x)}, \text{ then } f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{\left[h(x)\right]^2}$$

Book Notation: $\dfrac{d\left(\dfrac{g}{h}\right)}{dx} = \dfrac{\left(\dfrac{dg}{dx}\right).(h) - (g).\left(\dfrac{dh}{dx}\right)}{h^2}$

In Plain English, the derivative of the quotient of two functions (which we will call the "top" function and the "bottom" function) is equal to the **derivative of the top, times the bottom, minus the top, times derivative of the bottom, all over the bottom squared.**

Examples

A. $f(x) = \dfrac{(2x^3 + 8x)}{(5x^4 + 17)}$

We see that $f(x)$ consists of the quotient of two smaller functions, in this case $(2x^3+8x)$ "the top" and $(5x^4+17)$ "the bottom." So, the derivative then is:

$$f'(x)=\frac{\overbrace{(6x^2+8)}^{\textbf{Derivative of the top}}\times\overbrace{(5x^4+17)}^{\textbf{The bottom}}-\overbrace{(2x^3+8x)}^{\textbf{The top}}\times\overbrace{(20x^3)}^{\textbf{Derivative of the bottom}}}{\underbrace{(5x^4+17)^2}_{\textbf{The bottom squared}}}$$

Note: You should leave the answer in this form unless we are asked to "clean up."

Again, do not forget to label your derivative.

B. $g(x)=\dfrac{x}{\sin x}$

We see that $g(x)$ consists of the product of two smaller functions, in this case x "the top" and $\sin x$ "the bottom."

So, the derivative then is: $g'(x)=\dfrac{(1)\cdot\sin x-x\cdot\cos x}{[\sin x]^2}=\dfrac{\sin x-x\cdot\cos x}{\sin^2 x}$

More Examples: Find and LABEL the derivatives of each of the following functions:

6. (video) $f(x)=\dfrac{(12x^2+9\sqrt[3]{x})}{(5x^2-13)}$

7. **(video)** $f(x)=\dfrac{3x^5}{\cos x}$

8. $f(x)=\dfrac{\left(x^4-8x\right)}{(2x-1)}$

9. $g(x)=\dfrac{\sin x}{\cos x}$

10. **(video)** $f(x)=\left(\dfrac{\sin x}{3x+5}\right)\cdot\left(7x^2+2x\right)$

11. $l(x)=\dfrac{\sin x}{\sqrt{x}}\left(x^2+2x\right)$

12. Suppose $f\left(\dfrac{\pi}{6}\right)=7$ and $f'\left(\dfrac{\pi}{6}\right)=-5$, and let $g(x)=f(x)\cos x$ and $h(x)=\dfrac{\sin x}{f(x)}$

$g'\left(\dfrac{\pi}{6}\right)=$

$h'\left(\dfrac{\pi}{6}\right)=$

13. Use the table to find the value of the following

x	1	0	7	−2	−1
$f(x)$	−5	−1	−407	7	1
$g(x)$	−3	−2	−9	0	−1
$f'(x)$	−7	−2	−163	−10	−3
$g'(x)$	−1	−1	−1	−1	−1

a. $(fg)'(-1)=$

b. $\dfrac{f(-1)}{g(-1)+5}$

c. $(f+g)'(-1)=$

d. $(f-g)'(-1)=$

e. $(fg)'(7)=$

f. $\left(\dfrac{g}{f}\right)'(0)=$

14. Given that

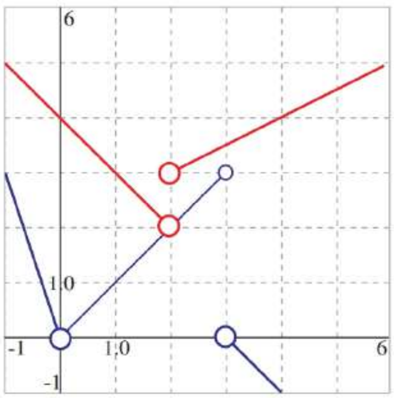
$f(x)=x^{12}h(x)$

$h(-1)=3$

$h'(-1)=6$

Calculate $f'(-1)$

15. The graphs of the function f (given in blue) and g (given in red) are plotted below. Suppose that $u(x) = f(x) \cdot g(x)$ and $v(x) = \dfrac{f(x)}{g(x)}$.



f is the bottom function

g is the top function

a. Find $u'(1)$

b. Find $v'(1)$

Section 2.5

Chain Rule

A. The Chain Rule

$$\left[h(g(x))\right]' = h'(g(x)) \cdot g'(x)$$

Alternative Notation

$f(x) = (h \circ g)(x) = h(g(x))$ then $f'(x) = \left[h'(g(x))\right] \cdot \left[g'(x)\right]$

$$\left[h(g(x))\right]' = \frac{d(h(g))}{dx} = \left(\frac{dh}{dg}\right) \cdot \left(\frac{dg}{dx}\right)$$

In Plain English, first, identify which function is on the "outside" and which is on the "inside." (For the composition $(f \circ g)(x) = f(g(x))$ we say that f is on the "outside" and g is on the "inside.") The derivative of this composition is equal to **the derivative of the outside (leave the inside alone) times the derivative of the inside**.

Examples

1. $f(x) = \sin(5x^5)$

First, let us identify which is on the "outside" and which is on the "inside": Here $\sin(\ldots)$ is the "outside" (i.e., "sin of something") and $5x^5$ is the "inside."

Derivative of the "outside" is $\cos(\ldots)$, and if we the leave the inside alone, this will be $\cos(5x^5)$

Derivative of the "inside" is $25x^4$

$$f'(x) = \underbrace{\left(\cos(5x^5)\right)}_{\textbf{Derivative of the outside (leave the inside alone)}} \quad \text{x} \quad \underbrace{\left(25x^4\right)}_{\textbf{Derivative of the inside}}$$

Note: This style of answer should only be "cleaned up" if you are given specific instructions to do (or if you have to compare it to a list of multiple choice answers)!
Again, do not forget to label your derivative

More Examples

2. $g(x) = \sin^2 x = [\sin x]^2$

We see that $g(x)$ consists of $[\cdots]^2$ as the "outside" and $\sin x$ as the "inside."
So, the derivative of the "outside" is $2[\ldots]$ and derivative of the "inside" is $\cos x$

$\Rightarrow \quad g'(x) = 2[\sin x] \cdot [\cos x]$

Examples: Find and LABEL the derivatives of each of the following functions:

1. **(video)** $f(x) = (x^2 + 3)^2$

2. **(video)** $h(x) = (x^3 + 5x + 7)^9$

3. **(video)** $g(x) = \sqrt{5x^2 + 11x - 2}$

4. **(video)** $k(x) = \sin(7x^3 + 3x - 5)$

5. **(video)** $m(x)=\dfrac{5}{\sqrt[3]{x^2+3\cos x+17}}$

6. $f(x)=\tan\left(6x^3\right)$

7. $k(x)=\tan(\sin x)$

8. $l(x)=\sqrt{x^2+1}$

9. $f(x)=\left(x^2+2x\right)^8$

10. $r(s)=\dfrac{1}{\sqrt[3]{s^4+s}}$

B. Combinations of Product, Quotient, and Chain Rule

In many problems, we need to use a combination of the Product, Quotient, and Chain Rule to find a derivative. Here we will work through lots of examples.

Examples: Find and LABEL the derivatives of each of the following functions. Do not clean up unless otherwise indicated.

11. $g(x)=\sin\left(x^2\right)\left(8x^3-1\right)$

12. $f(x)=\sin^3\left(x^2\right)$

13. $g(x)=\left(x-\cos^2 x\right)^4$

14. $r(\theta)=\dfrac{\cos(2\theta-4\pi)}{\theta^2}$

15. $a(x)=\left(x^2\sqrt{2x^3+1}\right)$

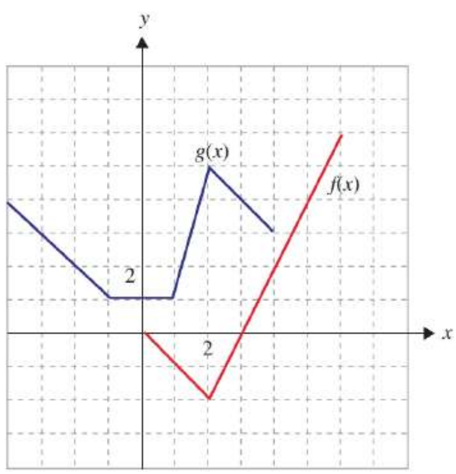
16. $f(x)=\dfrac{5x^3}{\left(x^2+5x\right)^3}$

17. If $f(x)=\cos(a^8+x^8)$, then $f'(x)=$

18. If $f(x)=\cos(x\sin x)$, find $f'(x)=$

19. Let $F(x) = f(g(x))$, where $f(-7) = 2, f'(-7) = 2, f'(3) = 15, g(3) = -7$, and $g'(3) = -8$, find $F'(3)$

20. If f and g are the functions whose graphs are shown below, let $u(x) = f(g(x))$ and $v(x) = g(f(x))$.



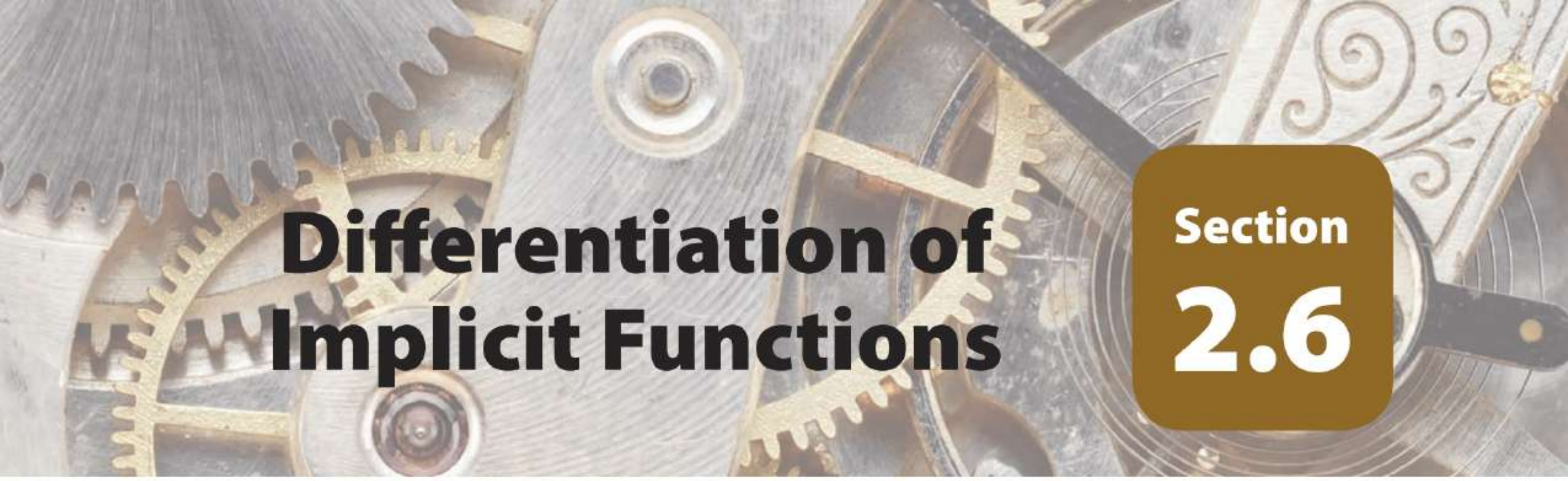
Find $u'(3)$

and $v'(3)$

21. Let $F(x) = f(x^6)$ and $G(x) = (f(x))^6$. You also know that $a^5 = 11, f(a) = 2, f'(a) = 8, f'(a^6) = 15$.

Find $F'(a)$

and $G'(a)$



Differentiation of Implicit Functions

Section **2.6**

Recall from the previous sections, that when we take the derivative of $y = f(x)$ then $y' = f'(x)$ where $f(x)$ is a function in terms of x (i.e., the only variable in the function is x).

Example: If $y = x^3 + 2x$ then $y' = \dfrac{dy}{dx} = 3x^2 + 2$

So, in other words, to take a derivative this way, we have to have the equation solved for "y."

Example: If $x = x^3 + 2x - y$. Here we first have to solve for y. So $y = x^3 + 2x - x \quad \Rightarrow \quad y = x^3 + x$, then $y' = \dfrac{dy}{dx} = 3x^2 + 1$.

Example: If $yx = x^3 + 2xy^2 - y$. Again, we have to solve for y in order to take a derivative in the way that we have learned in the preceding chapters. However, (as you can see in this case) it is not always easy/possible to do so.

HENCE: Implicit Differentiation!

A. Implicit Differentiation

Implicit Differentiation: Differentiation of a function where one variable (typically y) is not explicitly expressed as a function of another variable (typically x).

Here's how it works.

- It is important to pay attention to the notation. If we are given an equation in terms of x and y, and asked to find y' or $\dfrac{dy}{dx}$, we need to see that we are finding the derivative of $\boldsymbol{y}$, with respect to $\boldsymbol{x}$.
- We will treat both x and y like a variable, and take derivatives of each, but;
- When we take a derivative of a term containing "x" we will proceed as usual.
- When we take a derivative of a term containing "y" we will proceed as usual AND then also multiply the derivative of that term by $\dfrac{dy}{dx}$ (or y').
- We will use product, quotient, and chain rules as needed
- After differentiating, solve for (i.e., isolate) y' or $\dfrac{dy}{dx}$.

Example A: Find $\frac{dy}{dx}$ for $x = x^3 + y^2$.

$$\frac{d(x)}{dx} = \frac{d(x^3)}{dx} + \frac{d(y^2)}{dx} \quad \Rightarrow \quad 1 = 3x^2 + 2y \cdot \frac{dy}{dx}$$

Since we are trying to find $\frac{dy}{dx}$, isolate $\frac{dy}{dx}$ in our equation: $1 - 3x^2 = 2y \cdot \frac{dy}{dx} \quad \Rightarrow \quad \frac{1-3x^2}{2y} = \frac{dy}{dx}$

Example B: Find $\frac{dy}{dx}$ for $x = 4x^3 + y^2 - 8y$.

$$\frac{d(x)}{dx} = \frac{d(4x^3)}{dx} + \frac{d(y^2)}{dx} - \frac{d(8y)}{dx} \quad \Rightarrow \quad 1 = 12x^2 + 2y \cdot \frac{dy}{dx} - 8 \cdot \frac{dy}{dx}$$

Since we are trying to find $\frac{dy}{dx}$, isolate $\frac{dy}{dx}$ in our equation:

$$1 - 12x^2 = 2y \cdot \frac{dy}{dx} - 8 \cdot \frac{dy}{dx} \quad \Rightarrow \quad 1 - 12x^2 = \frac{dy}{dx}(2y - 8) \quad \Rightarrow \quad \frac{1-12x^2}{2y-8} = \frac{dy}{dx}$$

Example C: Find y' for $x = x^3y^2 - 3y^3$. (Notice that in this problem we have x^3y^2—a product of x and y. Here we will have to use the product rule.)

$$\frac{d(x)}{dx} = \frac{d(x^3y^2)}{dx} - \frac{d(3y^3)}{dx} \quad \Rightarrow \quad 1 = \left[(3x^2)(y^2) + (x^3)(2y \cdot y')\right] - 9y^2 \cdot y'$$

$$\Rightarrow \quad 1 - (3x^2)(y^2) = (x^3 \cdot 2y) \cdot y' - (9y^2) \cdot y' \quad \Rightarrow \quad 1 - (3x^2)(y^2) = y'(x^3 \cdot 2y - 9y^2)$$

$$\Rightarrow \quad \frac{1 - 3x^2y^2}{2x^3y - 9y^2} = y'$$

Examples: Find $\frac{dy}{dx}$ for the following:

1. **(video)** $x^2 + y - 11 = 8x$

2. **(video)** $x^2 + 3y^2 - 4x + 7y = 14$

3. **(video)** $x^3y^2+9y+3x=4$

4. **(video)** Find y' for $3y+12-x^2-y^4=0$ and evaluate at $(2,-1)$

5. $x^2+2y^2-11=0$

6. $y^2x-\dfrac{5y}{x+1}+3x=4$

7. Find y' for $2y+5-x^2-y^3=0$ and evaluate at $(2,-1)$

8. Find $\dfrac{dA}{dt}$ for $A=\pi r^2$

9. Find $\dfrac{dV}{dt}$ for $V=\dfrac{1}{3}\pi r^2 h$

10. If $f(x)+x^7\cdot[f(x)]^3=11$ and $f(2)=6$, find $f'(2)=$

11. Use implicit differentiation to find an equation for the tangent line to the curve $4x^2 - 4xy - y^3 = 84$ at the point $(1, -4)$.

Section 2.7

Related Rates

Before getting started with Related Rates, let us re-visit the following items first: Notation, Implicit Differentiation, and Geometric Formulas.

A. Notation

$$f'(x) \Leftrightarrow y' \Leftrightarrow \frac{dy}{dx}$$

Although all of the earlier notations are equivalent, we will use Leibniz's notation $\left(\frac{dy}{dx}\right)$ in this section, because it is more descriptive than the other forms. Leibniz's notation tells us specifically what we are taking a derivative of (in this case the function y) and what we are taking the derivative with respect to (w.r.t.)—that is, what is the variable in the function (in this case x.)

B. Implicit Differentiation

Again, we will have to pay close attention to notation here. In equations with multiple variables, we will be asked to find derivatives of specific parts of the equations with respect to specific variables (that may or may not be part of the equation!)

For Example: Consider the equation for the circle: $r^2 = x^2 + y^2$

- We would like to find $\frac{dr}{dt}$. This means, we are trying to find the derivative of ***r*** with respect to ***t***, that is, take a derivative of each term with respect to ***t***. (If a term is/contains a ***t***, just take a derivative as usual. If a term contains a variable other than ***t***, follow the usual rules for implicit differentiation.)

So, $(2r)\frac{dr}{dt} = (2x)\frac{dx}{dt} + (2y)\frac{dy}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{(2x)\frac{dx}{dt} + (2y)\frac{dy}{dt}}{2r}$

- We would like to find $\frac{dx}{dt}$.

So, $(2r)\frac{dr}{dt} = (2x)\frac{dx}{dt} + (2y)\frac{dy}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{(2r)\frac{dr}{dt} - (2y)\frac{dy}{dt}}{2x}$

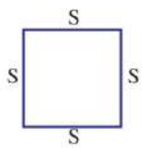
C. Geometric Formulas

You are responsible for knowing these formulas for all tests and the final exam

Two-Dimensional Shapes

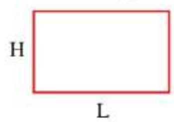
Shape — **Perimeter/ Circumference and Area**

Square



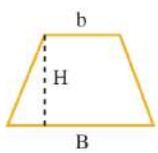
$$P = 4S$$
$$A = S^2$$

Rectangle



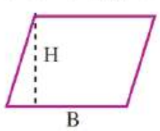
$$P = 2H + 2L$$

Trapezoid



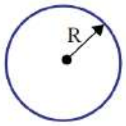
$$A = \frac{H(B+b)}{2}$$

Parallelogram



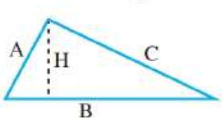
$$A = BH$$

Circle



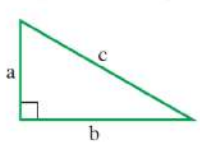
$$C = 2\pi R$$
$$A = \pi R^2$$

Triangle



$$A = \frac{BH}{2}$$

Right Triangle

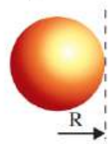


$$c^2 = a^2 + b^2$$

Three-Dimensional Shapes

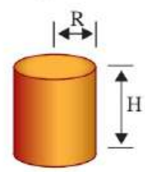
Shape

Sphere



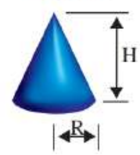
$$SA = 4\pi R^2$$
$$V = \frac{4}{3}\pi R^3$$

Cylinder



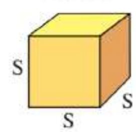
$$SA = 2\pi RH + 2\pi R^2$$
$$V = \pi R^2 H$$

Cone



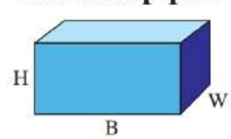
$$V = \frac{\pi R^2 H}{3}$$

Cube



$$SA = 6S^2$$
$$V = S^3$$

Rectangular Parallelepiped



$$SA = 2HB + 2BW + 2HW$$
$$V = BHW$$

D. Related Rate Problems

1. **(video)** Find $\frac{dV}{dr}$ for the function $V = \pi r^2 h$.

2. **(video)** Find $\frac{dV}{dt}$ for the function $V = \pi r^2 h$.

3. **(video)** Find $\frac{dA}{dh}$ for the function $A = \pi r^2$.

4. **(video)** Find $\frac{dA}{dt}$ for the function $A = \pi r^2$.

5. **(video)** If $z^2 = x^2 + y^2$, $\frac{dx}{dt} = 10$, and $\frac{dy}{dt} = -1$. Find the positive value of $\frac{dz}{dt}$ when $x = 5$ and $y = 12$.

6. Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1.5 m/s, how fast is the area of the spill increasing when the radius is 19 m?

Hint: The word **rate = derivative**. Pay attention to units to find out which rate is given asked for.

(Example, "rate of 1.5 m/s"—meters per second = unit of **length** per unit of **time** $= \frac{d(length)}{d(time)} = \frac{d(r)}{d(t)}$)

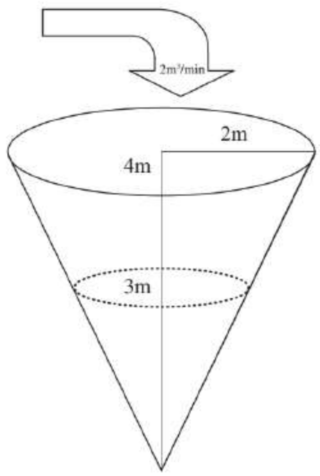
7. A fireman is on top of a 75-foot ladder that is leaning against a burning building. If someone has tided Sparky (the fire dog) to the bottom of the ladder and Sparky takes off after a cat at a rate of 6 ft/sec, then what is the rate of change of the fireman on top of the ladder when the ladder is 5 feet off the ground?

8. A street light is mounted at the top of a 11 ft tall pole. A woman 6 ft tall walks away from the pole with a speed of 8 ft/sec along a straight path. How fast is the tip of her shadow moving when she is 50 ft from the base of the pole?

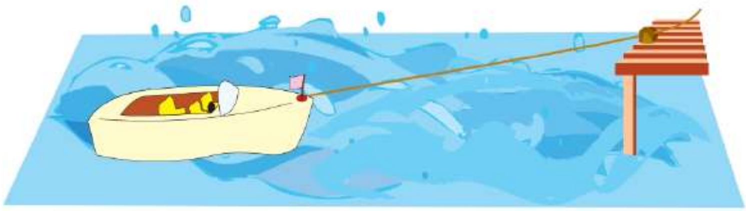
9. If a snowball melts so that its surface area decreases at a rate of $0.01\frac{cm^2}{\min}$, find the rate at which the diameter decreases when the diameter is 8 cm.

10. At noon, ship A is 30 miles due west of ship B. Ship A is sailing west at 25 mph and ship B is sailing north at 18 mph. How fast (in miles per hour) is the distance between the ships changing at 5 p.m.?

11. Water pours into in inverted cone at a rate of 2 m^3/min. If the cone has a radius of 2 m and a height of 4 m, find the rate at which the water level is rising when the water is 3 m deep.



12. A boat is pulled into a dock by means of a rope attached to a pulley on the dock. The rope is attached to the front of the boat, which is 7 ft below the level of the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 120 ft of rope is out?



Linear Approximation and Differentials

Section 2.8

A. Differentials

$$\text{Slope} = \frac{\Delta y}{\Delta x} \qquad \text{Slope of the tangent line} = \frac{dy}{dx} \qquad \Rightarrow \qquad \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

$$\text{Also, } \frac{dy}{dx} = f'(x) \;\Rightarrow\; dy = \left[f'(x)\right] \cdot dx$$

Example: Find the differential of y given that:

1. **(video)** $y = 5x^3 \cdot \sin x$

2. **(video)** $f(x) = \sqrt{5x^2 + 2x - 12}$

3. $y = 5x^3$

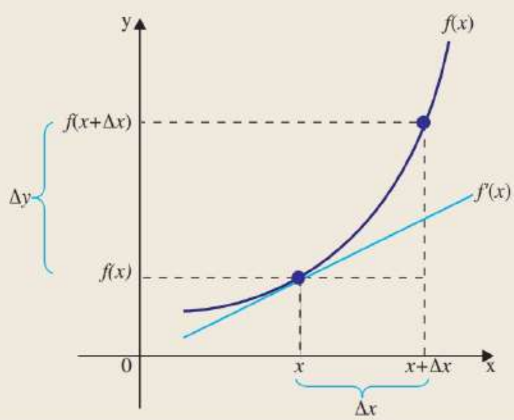
4. $y=\dfrac{3x^2+4x-5}{5\sin x+2x}$

Note:

$\Delta x \approx dx$ and $\Delta y \approx dy$

$dx \approx \Delta x = (x+\Delta x)-(x)$

$dy \approx \Delta y = f(x+\Delta x) - f(x)$



5. (video) A = πr^2 (Area of a circle)

a. Use differentials to approximate the change in area when going from $r = 4$ ft to $r = 4.2$ ft. (Find dA)

b. (video) Find the actual change in area when going from $r = 4$ ft to $r = 4.2$ ft. (Find ΔA)

6. **a.** If $y = x^3$ then calculate Δy for $x = 2$ to $x = 2.01$

b. Calculate dy for $x = 2$ to $x = 2.01$

7. $V = \frac{4}{3}\pi r^3$ (volume of a sphere)

a. Use differentials to approximate the change in volume when going from $r = 3\,ft$ to $r = 2.8\,ft$

(Find dV: Start by finding $\frac{dV}{dr}$)

b. Find the actual change in volume when going from $r = 3\,ft$ to $r = 2.8\,ft$

(Find ΔV)

B. Linearization

Definition: The linearization of a function $f(x)$ at a fixed point a is given by the formula

$$L(x)=f(a)+f'(a)(x-a)$$

Slope of the tangent line =
$m_T = f'(a)$

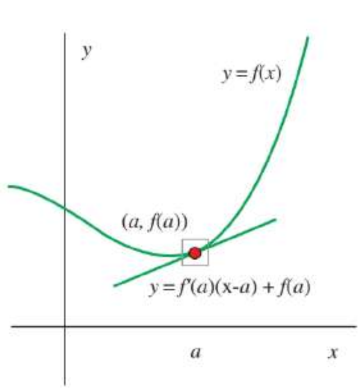
Point Slope:

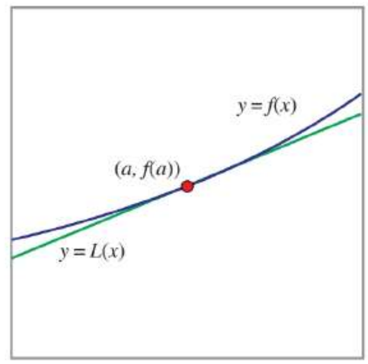
$(y-y_o)=m_T(x-x_0)$

$(y-f(a))=f'(a)(x-a)$

$y=f(a)+f'(a)(x-a)$

$L(x)=f(a)+f'(a)(x-a)$





Examples

8. **(video)** Find the linear approximation of $f(x)=\sqrt{x^2+3}$ at $a=2$

9. **(video)** Use linearization techniques to approximate $\sqrt{25.1}$

10. Find the linear approximation of $f(x) = \cos(5x)$ at $a = \dfrac{\pi}{2}$

11. Use linearization techniques to approximate $\sqrt{16.1}$

12. Find the linear approximation of $f(x)=\sqrt{4-x}$ at $a=0$ and use it to approximate $\sqrt{3.9}$ and $\sqrt{4.1}$.

13. Use a linear approximation to approximate 2.001^6 as follows:

The linearization $L(x)$ to $f(x)=x^6$ at $a=2$ can be written in the form $L(x)=mx+b$. Using this, the approximation for 2.001^6 is

14. The edge of a cube was found to be $60\ \text{cm}$ with a possible error of $0.5\ \text{cm}$. Use differentials to estimate the following:

a. The maximum possible error in the volume of the cube

b. The relative error in the volume of the cube

c. The percentage error in the volume of the cube

Exponential, Logarithmic, and Inverse Functions

Section **3R**

I. Review of Inverse Functions

A. Identifying One-to-One Functions

A function $f(x)$ is one-to-one if every element in the range corresponds to only one element in the domain.

If $f(a)=f(b)$ then $a=b$ or if $a\neq b$ then $f(a)\neq f(b)$

Horizontal Line Test: If there is NO horizontal line that intersects the graph more than once, then the function is one-to-one.

Example: Determine whether each function is one-to-one.

1. $f(x)=x^2+2$

2. $g(x)=\dfrac{x}{x-2}$

B. Inverse Functions

Let $f(x)$ be a function that is one-to-one and that goes through the point (a,b)

- Then $f^{-1}(x)$ is the inverse of $f(x)$
- $(f\circ f^{-1})(x)=x$
- $f^{-1}(x)$ will go through the point (b,a)
- The domain of $f(x)=$ the range of $f^{-1}(x)$
- The domain of $f^{-1}(x)=$ the range of $f(x)$

C. Finding Inverse Functions

Steps:
1. Test to see whether the function is one-to-one
2. Replace $f(x)$ with y
3. Interchange x and y
4. Solve equation for y
5. Replace y with $f^{-1}(x)$

Example: Verify that the functions are inverse of each other

1. $f(x)=\sqrt[3]{x+4}$ and $g(x)=x^3-4$

Example: Find the inverse for each of the following:

1. $k(x)=(x-1)^3$

2. $g(x)=\dfrac{2x-3}{x+1}$

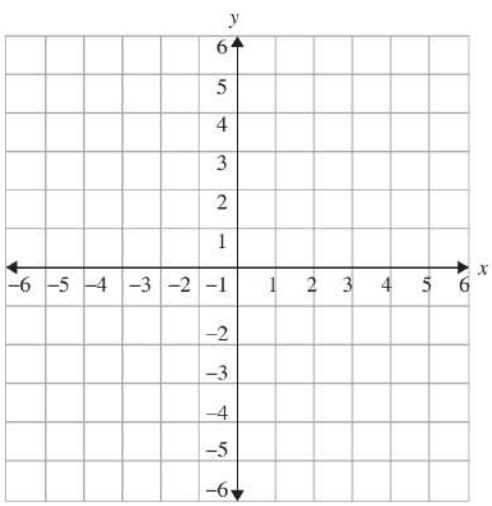
3. $h(x)=x^2+3$

D. Graphs of Inverse Functions

The graph of $f^{-1}(x)$ can be constructed by mirroring the graph of $f(x)$ over the line $y = x$

Examples

1. Construct the graph of $f^{-1}(x)$ if $f(x) = \sqrt{x}$



2. The following are points on the graph of $f(x)$: $(2, 10)$, $(-1, -3)$, $(0, -2)$, $(1, -1)$, $(3, 6)$

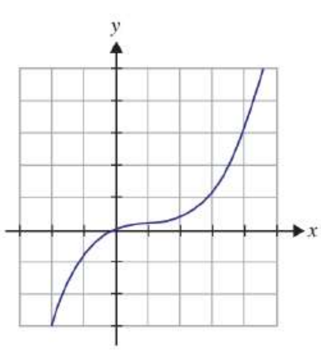
Find at least five points on the graph of $f^{-1}(x)$

D. Domain and Range of Inverse Functions

The domain of $f(x) =$ the range of $f^{-1}(x)$

The domain of $f^{-1}(x) =$ the range of $f(x)$

Example: A function $f(x)$ has the following graph. Find the domain and range of the inverse function $f^{-1}(x)$.



II. Review of Exponential and Logarithmic Functions

A. Exponential Functions

DEFINITION: An exponential function is a function in the form $f(x) = a^x$. (i.e., the variable x is in the exponent)

Example: Find x for each of the following:

1. $4 = 2^x$ **2.** $27 = 3^x$ **3.** $2 = 4^x$

B. Logarithmic Functions

I. Logarithmic Functions

A logarithm is a function that helps us to solve a quadratic function/logarithms allow us to isolate the variable in a quadratic function (and the other way around).

DEFINITION: A logarithmic function is a function in the form $f(x) = \log_a x$ (i.e., the variable x is in the expression) $\boldsymbol{y = \log_b x}$ "***y*** is equal to log base ***b*** of ***x***" - Here "***b***" is the BASE NUMBER and "***x***" is the VARIABLE.

$\log_b x = y$ means exactly the same thing as $b^y = x$

Examples: Write each equation in its equivalent form:

1. $x = \log_2 16$ **2.** $y = \log_6 216$

3. $3 = \log_b 27$ **4.** $8^y = 300$

II. Common Logarithmic Properties

1. $\log_b b = 1$	**4.** $\log_b b^x = x$
2. $\log_b 1 = 0$	**5.** $b^{\log_b x} = x$
3. $\log_b 0 = DNE$	**6.** $\log x = \log_{10} x$

Example: Simplify Each Expression

1. $\log_2 2 =$

2. $\log_6 1 =$

3. $\log_4 4^x =$

4. $\log_z z^{(x+y)} =$

5. $8^{\log_8(12y)} =$

6. $\log 10 =$

III. The Natural Logarithm

DEFINITION: e is a number the equals approximately 2.718281828 $\quad$ $\boldsymbol{\log_e x = \ln x}$

Example

1. $\log_e(3z) =$

2. $\ln e =$

IV. Expansion Properties for Logarithms

1. $\log_b(M \cdot N) = \log_b M + \log_b N$ (Product Rule)

2. $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$ (Quotient Rule)

3. $\log_b(M^n) = n \cdot \log_b M$ (Power rule)

Example: Simplify the following:

1. $\log_x(xz) =$

2. $\log_3 27 =$

3. $2^{\log_2(5y)} =$

4. $\log 10^{(x+y)} =$

Expand the following logarithms

5. $\log_b\left((xy)^3\right)$

6. $\log_b\left(xy^3\right)$

7. $\log_b\left(\dfrac{x^3y}{z^2}\right)$

8. $\ln\left(\sqrt{ex}\right)$

9. $\ln\left(\dfrac{x^4\sqrt{x^2+3}}{(x+3)^5}\right)$

10. $\log\left(\dfrac{100x(x+3)}{z(x-4)^3}\right)$

Write the following as single logarithms:

11. $3\log_b z + 4\log_b x$

12. $\frac{1}{2}\log(x+2) - \log x$

13. $3\log_5 z - \frac{1}{2}\log_5(z+2) + 2\log_5 y$

V. Change of Base Formula for Logarithms

$$\log_b M = \frac{\log M}{\log b} = \frac{\ln x}{\ln b}$$

Example: Use your calculator to find:

1. $\log_8 17$

2. $\log_5 15$

C. Solving Exponential and Logarithmic Functions

I. Common Base Property for Exponential Functions

If $b^M = b^N$, then $M = N$

Example

1. Solve $3^{2x} = 3^{x-5}$

2. Solve $8^x = 2^{x+4}$

II. "Exponentiating" (How to solve equations involving e and ln)

$$\ln b^x = x \cdot \ln b$$
$$\ln e^x = x$$

Example

1. Find x if $e^{7x+3} = 5$

2. Find x if $2^x = 12$

III. Common Base Property for Logarithmic Functions

If $\log_b M = \log_b N$, then $M = N$

Example: Solve $\ln(x+2) - \ln(4x+3) = \ln\left(\frac{1}{x}\right)$

IV. Solving for a variable in the exponent

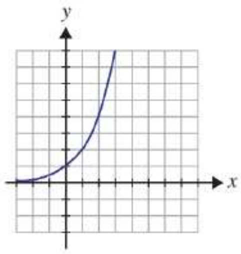
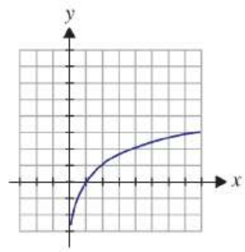
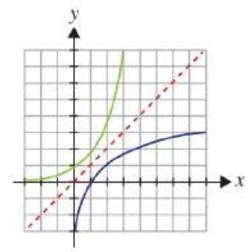
Example

1. $R = 25 \cdot e^{38 \cdot x}$

2. $y = 8 \cdot e^{t \cdot x}$

D. Graphs of Exponential and Logarithmic Functions

I. Comparison of Logarithmic function graph to Exponential function graph

$y = 2^x$	$y = log_2 x$	**Comparison of the two graphs, showing the inversion line in red**
		

If $f(x) = a^x$ and $g(x) = \log_a x$ then $f(x)$ and $g(x)$ are inverses of each other.

$f(x) = a^x$	Domain: All x (No Restrictions)	$(-\infty, \infty)$
	Range: $y > 0$	$(0, \infty)$
$f(x) = \log_a x$	Domain: $x > 0$	$(0, \infty)$
	Range: All x (No Restrictions)	$(-\infty, \infty)$

*Note: Since exponential and logarithmic functions (with the same variable and base number) and are inverses of each other, the domain of one is the range of the other and vice versa.

Example

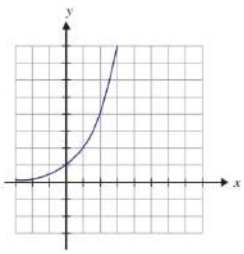
Find the domain and range of the following:

1. $f(x) = \log_5 x$ **2.** $f(x) = \log_5(x-4)+2$ **3.** $f(x) = 5^x + 5$

Exponential Functions

Section 3.1

A. Graph of an Exponential Function



B. Limit Rules

1. $\lim_{r \to x} a^r = a^x$
2. If $a > 1$, then $\lim_{x \to \infty} a^x = \infty$ and $\lim_{x \to -\infty} a^x = 0$
3. If $0 < a < 1$, then $\lim_{x \to \infty} a^x = 0$ and $\lim_{x \to -\infty} a^x = \infty$
4. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
5. $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to -\infty} e^x = 0$

Examples

1. **(video)** Solve the equation $3^x = \dfrac{1}{9}$
2. **(video)** Solve the equation $8^{4x+1} = 3$

3. Starting with the graph of $f(x)=9^x$, write the equation of the graph that results from:

a. Shifting $f(x)$ 9 units upward

b. Shifting $f(x)$ 7 units to the right

c. Reflecting $f(x)$ about the x-axis

4. Give the domain of the function:

a. $f(x)=\dfrac{16}{1-e^x}$

b. $g(x)=\dfrac{16}{1+e^x}$

5. Find the exponential function $f(x)=Ca^x$ whose graph goes through the points (0, 5) and (2, 20).

6. **(video)** Find $\lim\limits_{x\to\infty} 3e^{-2x}$

7. (**video**) Find $\lim_{x\to\infty} \frac{3e^x}{5e^x + 12}$

8. Evaluate $\lim_{x\to\infty} 0.77^x =$

9. Evaluate $\lim_{x\to\infty} e^{-x^2} =$

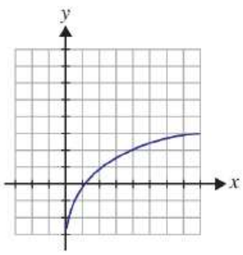
10. Evaluate $\lim_{x\to\infty} \frac{e^{5x} + e^{-5x}}{e^{5x} - e^{-5x}} =$

11. Evaluate $\lim_{x\to\infty} \left(e^{-2x} \cos x\right) =$

Logarithmic Functions

Section
3.2

A. Graph of a Logarithmic Function



A. Review

Expansion Properties for Logarithms

4. $\log_b (M \cdot N) = \log_b M + \log_b N$ (Product Rule)

5. $\log_b \left(\dfrac{M}{N}\right) = \log_b M - \log_b N$ (Quotient Rule)

6. $\log_b \left(M^n\right) = n \cdot \log_b M$ (Power rule)

Examples

1. **(video)** For the function $f(x) = \dfrac{3x+1}{2x+4}$ the inverse function $f^{-1}(x)$ equals

2. **(video)** Expand the expression $\log\left(\dfrac{1000\, x^4 \sqrt{z}}{(y+5)^2}\right)$ completely: log

3. **(video)** Write $4\log x+\frac{1}{2}\log z-2\log(y+3)+\log 8$ as a simple logarithm and simplify as much as possible.

B. Limit Rules

6. $\lim_{x\to 0^+}\ln x=-\infty$

7. $\lim_{x\to\infty}\ln x=\infty$

8. $\lim_{x\to 0^+}\log x=-\infty$

9. $\lim_{x\to\infty}\log x=\infty$

Examples

4. **(video)** Find $\lim_{x\to\infty}\frac{-2\ln x+3}{5}$

5. **(video)** Find $\lim_{x\to 0^+}\frac{\ln x+3x-2}{4x^2-3}$

6. $\lim_{x\to 3^+} \log\left(x^2-5x+6\right)=$

7. $\lim_{x\to 0^+} \log(\sin x)=$

8. $\lim_{x\to\infty}\left[\log\left(1+x^2\right)-\log(1+x)\right]=$

9. If f is one-to-one and $f(3)=11$, then

a. $f^{-1}(11)=$

b. $[f(3)]^{-1}=$

10. Find the inverse for each of the following:

a. $f(x)=\dfrac{4x-12}{19x+15}$

b. $h(x)=e^{9x+3}$

c. $f(x)=\ln(13x+10)$

11. If $f(x) = x^3 + 1$

a. Find $f^{-1}(x) =$

b. Find $\left(f^{-1}\right)'(x) =$

c. Find $\left(f^{-1}\right)'(2) =$

C. "Derivative of Inverse" Formula

When we cannot find $f^{-1}(x)$ and $f(x)$ is one-to-one: $\left(f^{-1}\right)'(a) = \dfrac{1}{f'\left(f^{-1}(a)\right)}$

12. For $f(x) = x^3 + 4x + 4$, find $\left(f^{-1}\right)'(4) =$

13. If $f(x)=\sqrt{x^3+x^2+x+1}$ find $\left(f^{-1}\right)'(1)=$

14. Suppose f^{-1} is the inverse function of a differentiable function f and $f(3)=4$, $f'(3)=\dfrac{7}{4}$ then $\left(f^{-1}\right)'(4)=$

15. If $\ln a=2$, $\ln b=3$, and $\ln c=5$, evaluate $\ln\left(\sqrt{b^{-4}c^{-4}a^{-3}}\right)=$

16. Solve each equation for x:

a. $3^{x-4}=9$

b. $\ln x+\ln(x-1)=4$

Derivatives of Exponential and Logarithmic Functions

Section
3.3

A. Derivatives

1. $\dfrac{d}{dx}\left(\log_a x\right) = \dfrac{1}{x \cdot \ln a}$
2. $\dfrac{d}{dx}\left(\ln x\right) = \dfrac{1}{x}$
3. $\dfrac{d}{dx}\left(a^x\right) = a^x \cdot \ln a$
4. $\dfrac{d}{dx}\left(e^x\right) = e^x$

Examples: Find the derivative of each

1. **(video)** $f(x) = 5\log_7 x$

2. **(video)** $f(x) = 17e^x + 5x - \sin x$

3. **(video)** $y = e^{-5\tan x}$

4. (video) $y = e^{-5x} \cdot \cos(3x)$

5. (video) $f(x) = \ln x$

6. (video) $f(x) = 3x^6 \cdot \ln(3x^2 + 4)$

7. $f(x) = \ln\left(x^3 + 10\right)$

8. $f(x)=\ln\left(\sqrt{\dfrac{4x-6}{x+1}}\right)$

9. $f(x)=5\sqrt{x}\cdot e^{2x}$

10. $f(x) = \log_3\left(5x^2 + 4\right)$

11. $f(x) = 7^{5x}$

B. Logarithmic Differentiation

Examples: Find the derivative of each

12. $y = x^{\sqrt{x}}$

13. $y = \dfrac{(9x+4)^7\left(x^3-5\right)^3}{\sqrt{3x-1}}$

Exponential Growth and Decay

Section

3.4

A. Population Growth

$$P(t) = P(0) \cdot e^{kt}$$

where

$P(t)$ = Population after t years

$P(0)$ = Initial Population

K = Growth constant

t = Time

Examples

1. **(video)** Solve $P(t) = P_0 \cdot e^{kt}$ for t.

2. **(video)** Solve the equation $9{,}000 = 1{,}500e^{kt}$ for t.

3. A bacteria culture initially contains 600 cells and grows at a rate proportional to its size. After 5 hours the population has increased to 620.

   a. Find an expression for the number of bacteria after t hours.

b. Find the number of bacteria after 7 hours.

c. Find the rate of growth after 7 hours. (Remember: Rate = Derivative)

d. When will the population reach 4,000?

B. Half-Life

$$P(t) = P(0) \cdot e^{kt}$$

where

$P(t)$ = Quantity after t years

$P(0)$ = Initial Quantity

K = Decay constant

t = Time

4. The half-life of cesium-137 is 30 years. Suppose we have a 900-mg sample.

a. Find the mass that remains after t years. (Find an expression for the mass that remains after t years.)

b. How much of the sample remains after 150 years?

c. After how long will only 4 mg remain?

d. Find the rate of decay after 20 years.

C. Newton's Law of Cooling

	Alternatively
$$T(t)=(T_0-T_s)\cdot e^{kt}+T_s$$	$$T(t)=C\cdot e^{kt}+T_s$$
where	where
$T(t)$ = Temperature after time t	$T(t)$ = Temperature after time t
T_s = Temperature of surrounding area	T_s = Temperature of surrounding area
T_0 = Initial temperature of object	C = Initial temperature − surrounding temperature
K = Growth constant	K = Growth constant
t = Time	t = Time

5. A roast turkey is taken from an oven when its temperature has reached 175 Fahrenheit and is placed on a table in a room where the temperature is 65 Fahrenheit.

a. If the temperature of the turkey is 155 Fahrenheit after half an hour, what is its temperature after 45 minutes?

b. When will the turkey have cooled to 110 Fahrenheit?

D. Interest

Compound Interest

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where

A = Future Value

P = Initial Value

r = Interest rate

n = Number of times per year compounded

t = Time in years

Continuous Interest

$$A = P \cdot e^{rt}$$

where

A = Future Value

P = Initial Value

r = Interest rate

t = Time in years

6. **(video)** Find the accumulated value of an investment of \$120,000 at an interest rate of 4.5% if it is compounded monthly for 5 years.

7. **(video)** Find the accumulated value of an investment of \$120,000 at an interest rate of 4.5% if it is compounded continuously for 5 years.

8. If 8,000 dollars is invested at 9% interest, find the value of the investment at the end of 5 years if interest is compounded

a. Annually

b. Quarterly

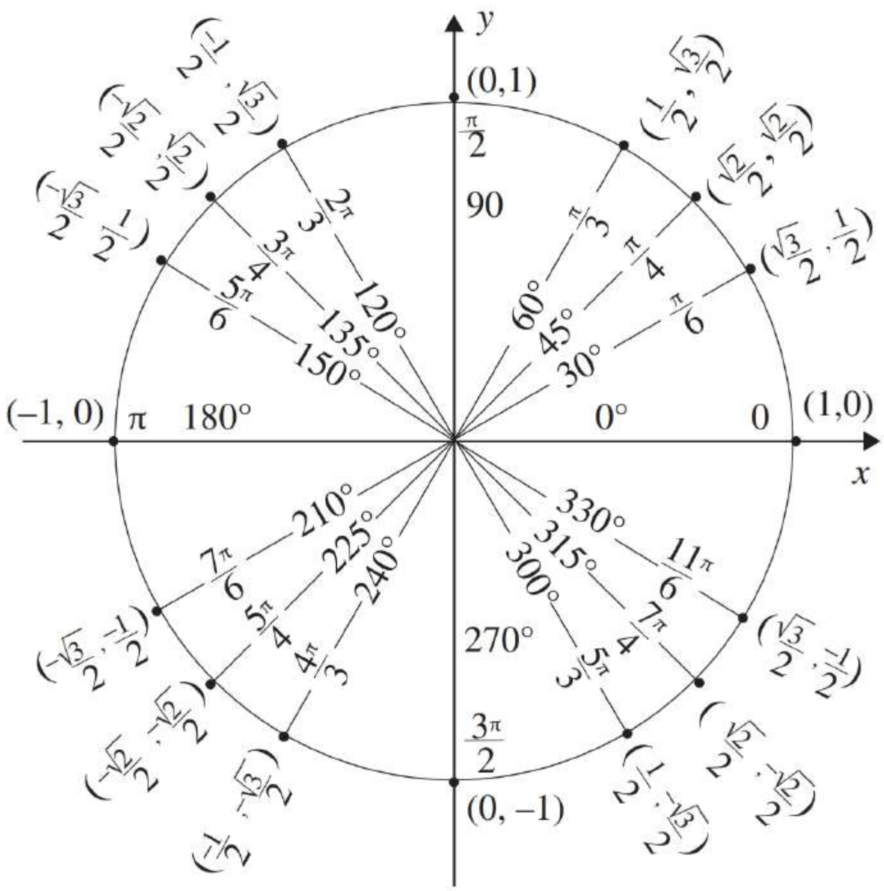
c. Monthly

d. Continuously

Inverse Trigonometric Functions

Section 3.5

A. Unit Circle and Common Values



Degree	**Radians**	**Sin θ**	**cos θ**	**tan θ**	**cot θ**	**sec θ**	**csc θ**
0°	0	0°	1	0°	Undefined	1	Undefined
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
45	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
90°	$\pi/2$	1	0	Undefined	0	Undefined	1
120°	$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
135°	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$5\pi/6$	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2
180°	π	0	-1	0	Undefined	-1	Undefined
210°	$7\pi/3$	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2
225°	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$
270°	$3\pi/2$	-1	0	Undefined	0	Undefined	-1
300°	$5\pi/2$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-\sqrt{3}$	2	$-2\sqrt{3}/3$
315°	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$11\pi/6$	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2
360°	2π	0	1	0	Undefined	1	Undefined

Examples

1. (Video) Give the exact value of $\cos^{-1}\left(\frac{1}{2}\right)$

2. (Video) Give the exact value of $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

3. **(Video)** Give the exact value of $\sin^{-1}\left(\frac{1}{2}\right)$

4. **(Video)** Give the exact value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

5. **(Video)** Give the exact value of $\tan^{-1}\left(\sqrt{3}\right)$

6. **(Video)** Give the exact value of $\tan^{-1}(1)$

7. **(Video)** Find $\lim_{x\to\infty} \tan^{-1} x$

8. **(Video)** Find $\lim_{x\to-\infty} \tan^{-1} x$

B. Derivatives of Inverse Trigonometric Functions (You Must Know These!)

1. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$	**4.** $\dfrac{d}{dx}\left(\csc^{-1} x\right) = -\dfrac{1}{x\sqrt{x^2-1}}$
2. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$	**5.** $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$
3. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$	**6.** $\dfrac{d}{dx}\left(\cot^{-1} x\right) = -\dfrac{1}{1+x^2}$

Examples

9. **(video)** Find $\dfrac{d}{dx}\left(5\sec^{-1}x+2x^2+3\right)$

10. **(video)** Find $\dfrac{d}{dx}\left(5\cos^{-1}\left(3x^2+5\right)\right)$

11. **(video)** Find $\dfrac{d}{dx}\left(\tan^{-1}\left(5x^2+17\right)\right)$

12. Let $f(x)=\left(\tan^{-1}x\right)^7$. Find $f'(x)$.

13. Let $f(x)=\cos^{-1}\left(e^{8x}\right)$. Find $f'(x)$.

14. Find the limit: $\lim_{x\to 1^-} \sin^{-1} x$

15. Find the limit: $\lim_{x\to\infty} \arctan\left(-e^x\right)$

Indeterminate Forms and L'Hopital's Rule

Section 3.7

Suppose that $f(x)$ and $g(x)$ are differentiable, $g'(x) \neq 0$ and that $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or that $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \pm\frac{\infty}{\infty}$ (i.e., we have an in determinant form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

A. Indeterminate form $\frac{\infty}{\infty}$ or $\frac{0}{0}$

If we have a limit of the form $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to 0$ and $g(x) \to 0$, then we have the in determinant form of type $\frac{0}{0}$

If we have a limit of the form $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to \infty$ and $g(x) \to \infty$, then we have the in determinant form of type $\frac{\infty}{\infty}$.

Examples

1. **(video)** $\lim\limits_{x \to 2} \frac{2x^2 - 8}{x - 2}$

2. **(video)** $\lim\limits_{x \to 0} \frac{3e^{2x} - 3}{x}$

3. **(video)** $\lim_{x\to 1}\frac{3-3x}{5\ln x}$

4. **(video)** $\lim_{x\to\infty}\frac{x^2+x+3}{2x^2+5}$

5. **(video)** $\lim_{x\to\infty}\frac{x+3}{\sqrt{x}+1}$

6. $\lim_{x\to -2}\frac{x^2-4}{x+2}$

7. $\lim_{x\to 0}\frac{1-e^{ax}}{x^7}$, assume $a>0$

8. $\lim_{x\to 0}\frac{\sin(10x)}{\sin(bx)}$

9. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2}$

10. $\lim\limits_{x \to 1} \dfrac{\ln x}{x-1}$

11. $\lim\limits_{x \to 0} \dfrac{\sin x}{1-\cos x}$

12. $\lim\limits_{x \to 0} \dfrac{a^x - 5^x}{9x}$

13. $\lim\limits_{x \to 0} \dfrac{1+x-e^x}{3x^2}$

B. Indeterminate form $\infty \cdot 0$

Hint: Remember, in order to use L'Hopital's Rule the expression must be in the form $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$

If not, our first step is to get it in that form.

If $\lim\limits_{x\to a} f(x)\cdot g(x)$ equals either $0\cdot\infty$ or $\infty\cdot 0$ then you will need to rewrite it first.

You could either rewrite it as $\lim\limits_{x\to a}\dfrac{f(x)}{\left(\dfrac{1}{g(x)}\right)}$ or $\lim\limits_{x\to a}\dfrac{g(x)}{\left(\dfrac{1}{f(x)}\right)}$

*Try putting the EASY function on the bottom!

Examples

14. $\lim\limits_{x\to 0^+} x^4 \ln x$

15. $\lim\limits_{x\to 0} \cot(2x)\cdot\sin(6x)$

16. $\lim\limits_{x\to \infty} x^5 e^{-x^4}$

C. Other Indeterminate Forms

1. $\infty - \infty$
2. 0^0
3. ∞^0
4. 1^∞

Examples of the form "$\infty - \infty$"

17. $\lim\limits_{x \to 0^+} \csc(ax) - \cot(ax)$

Examples of the form "0^0, ∞^0, or 1^∞"

18. $\lim\limits_{x \to 0^+} x^x$

19. $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^{3x} =$

20. $\lim_{x \to 0^+} (1 - 7x)^{\frac{1}{x}} =$

Maximum and Minimum Values

Section 4.1

A. Absolute Maximum or Minimum/Extreme Values

A function $f(x)$ has an **Absolute Maximum** at $x = c$ if $f(c) \geq f(x)$ for every point x in the domain.

Similarly, $f(x)$ has an **Absolute Minimum** at $x = c$ if $f(c) \leq f(x)$ for every point x in the domain.

B. Local/Relative Maximum or Minimum Values

A function $f(x)$ has a **Local Maximum** at $x = c$ if $f(c) \geq f(x)$ for every point x that is near c.

A function $f(x)$ has a **Local Minimum** at $x = c$ if $f(c) \leq f(x)$ for every point x that is near c.

C. The Extreme Value Theorem

If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains both a maximum and a minimum value on $[a,b]$.

D. Fermat's Theorem

If $f(x)$ has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

E. Critical Number

A **critical number** of a function $f(x)$, is a number c in the domain such that $f'(c)=0$ or $f'(c)$ DNE (does not exist).

If $f(x)$ has a local maximum or minimum at c, then c is critical number of $f(x)$.

E. Closed Interval Method

To find **Absolute Maximum or Minimum** of a continuous function $f(x)$ on a closed interval $[a,b]$:

1. Find the values of $f(x)$ at the critical numbers of $f(x)$ in (a,b).
2. Find the values of $f(x)$ at the endpoints a and b of the interval.
3. The largest of the values of step 1 and 2 is the **Absolute Maximum**
4. The smallest of the values of step 1 and 2 is the **Absolute Minimum**

***The Closed Interval Method is used to find all absolute Maximums and absolute Minimums of a function over a closed interval!**

Examples

1. **(video)** The *critical number* of the function $f(x)=2x^2+12x-3$ is

2. **(video)** The *critical numbers* of the function $f(x)=3x^3-9x^2+15$ is

3. (video) The *critical numbers* of the function $f(x) = x^{\frac{4}{5}}(x+3)$ is

4. (video) Consider the function $f(x) = 4x^2 - 2x + 10$ on the interval $0 \le x \le 6$.

The absolute maximum value of $f(x)$ (on the given interval) is __________
and this occurs at x equals__________

and the absolute minimum of $f(x)$ (on the given interval) is__________
and this occurs at x equals__________

5. Find the critical numbers for the following functions: $f(x) = 2x^2 + 9x - 2$

6. Find the critical numbers for the following functions: $f(x) = -2x^3 + 33x^2 - 60x + 11$

7. Find the critical numbers for the following functions: $f(x) = x^{\frac{4}{5}} \cdot (x-5)$

8. Find the critical numbers for the following functions: $f(x) = (6x-2)e^{-6x}$

9. Consider the function $f(x) = 3x^2 - 6x + 8,\ 0 \le x \le 10$.

The absolute maximum value of $f(x)$ (on the given interval) is __________
and this occurs at x equals__________

and the absolute minimum of $f(x)$ (on the given interval) is__________
and this occurs at x equals__________

10. Consider the function $f(x) = 2x^3 + 18x^2 - 162x + 9$, $-9 \le x \le 4$.

The absolute maximum value of $f(x)$ (on the given interval) is __________
and this occurs at x equals__________

and the absolute minimum of $f(x)$ (on the given interval) is__________
and this occurs at x equals__________

11. Consider the function $f(x) = x + 2\cos x$ on the interval $0 \le x \le \pi$. Find the absolute maximum and minimum of the function.

12. Choose the best reason that the function $f(x) = x^{91} + x^{25} + x^7 + 13x + 2$ has neither a local maximum nor a local minimum.

a. The function $f(x)$ is always positive.

b. The derivative $f'(x)$ is always negative.

c. The derivative $f'(x)$ is always positive.

d. The highest power of x in $f(x)$ is odd.

The Mean Value Theorem

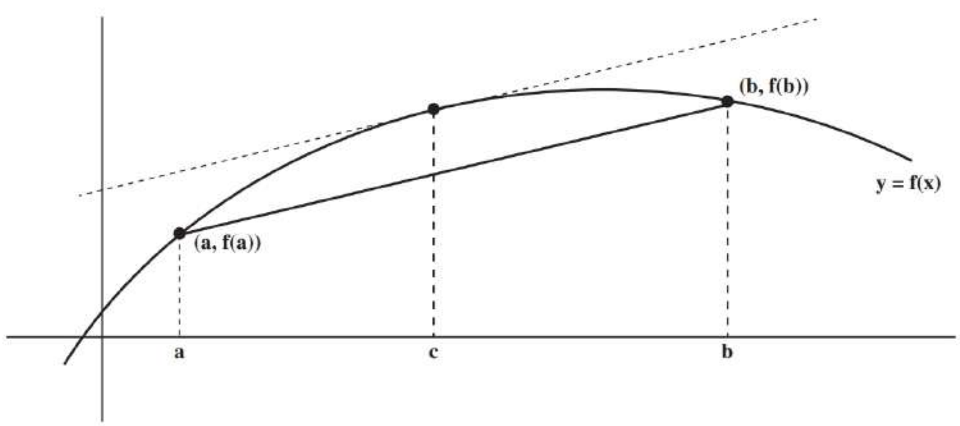
Section **4.2**

A. Rolle's Theorem

Let $f(x)$ be a function such that $f(x)$ is continuous on $[a,b]$, $f(x)$ is differentiable on (a,b) and $f(a)=f(b)$, then there is a number c in (a,b) such that $f'(c)=0$

B. The Mean Value Theorem

Let $f(x)$ be a function such that $f(x)$ is continuous on $[a,b]$ and $f(x)$ is differentiable on (a,b), then there is a number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$, or equivalently $f(b)-f(a)=f'(c)\cdot(b-a)$



C. Constant Theorem

If $f'(x)=0$ for all x in an interval (a,b) then $f(x)$ constant on (a,b).

D. Corollary

If $f'(x)=g'(x)$ for all x in an interval (a,b) then $f-g$ is constant on (a,b)
(i.e., $f(x)=g(x)+c$)

Examples

1. **(video)** Consider the function $f(x)=3x^2-6x+4$ on the interval [−1,3]. Verify that this function satisfies the three hypotheses of Rolle's Theorem on the interval.

   $f(x)$ is____________________on [−1, 3]

   $f(x)$ is____________________on [−1, 3]

   and $f(-1)=f(3)=$ ____________

   Then by Rolle's theorem, there exists a c such that $f'(c)=0$. Find the value c.

2. **(video)** Consider the function $f(x)=3x^2+x+4$ on the interval [−1, 4]. Find the average or mean slope of the function on this interval, that is,

   $$\frac{f(4)-f(-1)}{4-(-1)}=$$

   By the Mean Value Theorem, we know there exists a c in the open interval (−1, 4) such that $f'(c)$ is equal to this mean slope. For this problem, there is only one c that works.
   $c=$ ________

3. Consider the function $f(x) = x^2 - 4x + 1$ on the interval $[0,4]$. Verify that this function satisfies the three hypotheses of Rolle's Theorem on the interval.

$f(x)$ is_______________on $[0,4]$; $f(x)$ is_________________on $(0,4)$; and $f(0) = f(4) =$ _____________

Then by Rolle's Theorem, there exists a number c such that $f'(c) = 0$. Find the value c.

4. Consider the function $f(x) = 3x^2 + 5x + 11$ on the interval [–3, 6]. Find the average or mean slope of the function on this interval, that is, $\dfrac{f(6) - f(-3)}{6 - (-3)}$

By the Mean Value Theorem, we know there exists a c in the open interval (–3, 6) such that $f'(c)$ is equal to this mean slope. For this problem, there is only one c that works. $c =$_______

5. By applying Rolle's Theorem, check whether it is possible that the function $f(x) = x^5 + x - 11$ has two real roots.

Possible or impossible?

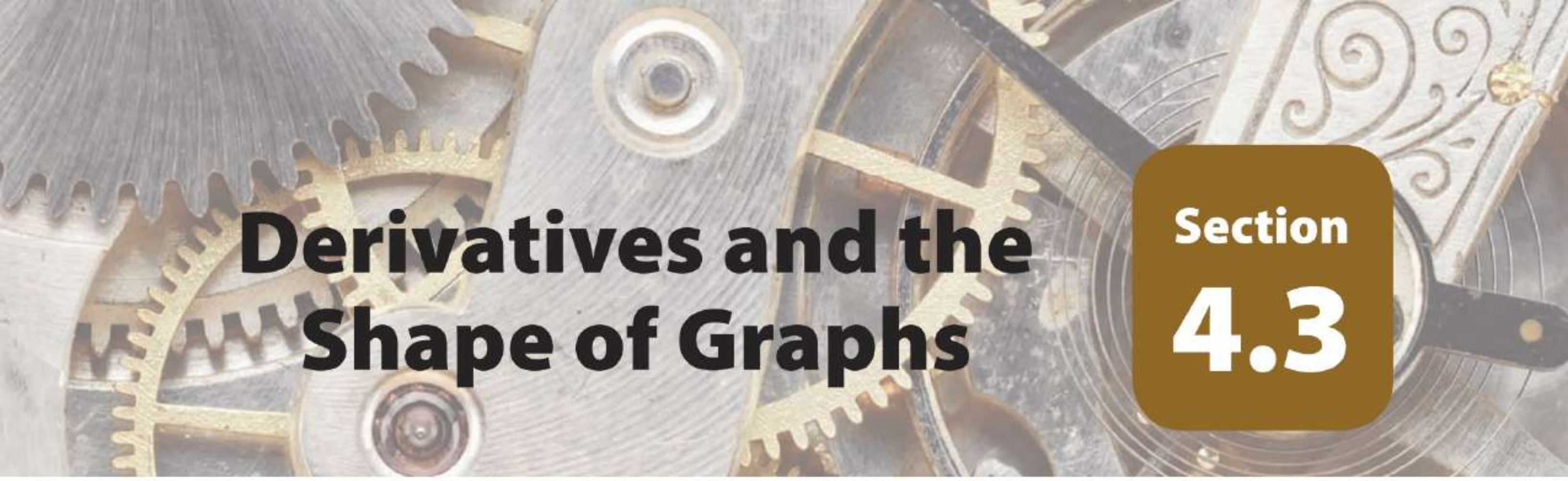
Your reason is that if $f(x)$ has two real roots then by Rolle's Theorem: $f'(x)$ must be

____

at certain value of x between these two roots, but $f'(x)$ is always negative, positive, or zero_____________

6. Suppose $f(x)$ is continuous on [2, 8] and $3 \le f'(x) \le 9$ for all x in (2, 8). Use the Mean Value Theorem to estimate $f(8) - f(2)$.

___ $\le f(8) - f(2) \le$ ___



Derivatives and the Shape of Graphs

Section 4.3

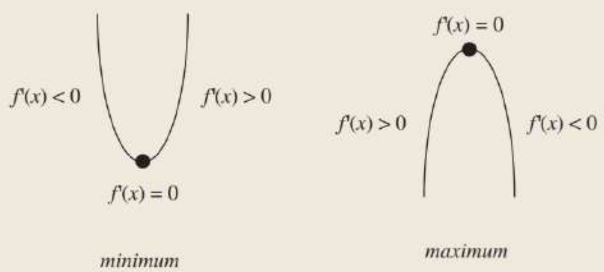
A. The First Derivative

Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

A **critical number** of a function $f(x)$, is a number c in the domain such that $f'(c) = 0$ or $f'(c)$ DNE.

If $f(x)$ has a local maximum or minimum at c, then c is critical number of $f(x)$.



The First Derivative Test

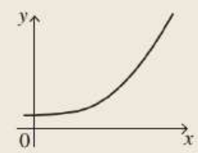
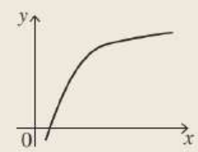
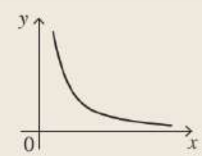
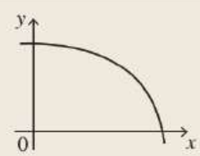
Suppose c is a critical number of a continuous function $f(x)$

- If $f'(x)$ changes from positive to negative at c, then $f(x)$ has a local maximum at c.
- If $f'(x)$ changes from negative to positive at c, then $f(x)$ has a local minimum at c.
- If $f'(x)$ does not change sign at c, then $f(x)$ has no local maximum or minimum at c.

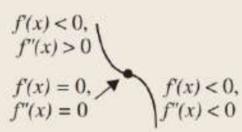
B. The Second Derivative

Concavity

- If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval.
- If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.

	Concave Up	Concave Down
Increasing Slope		
Decreasing Slope		

An **inflection point** of a function $f(x)$ is a point at which the curvature (second derivative) changes sign. The curve changes from being concave upward (positive curvature) to concave downward (negative curvature), or vice versa.



inflection point

The Second Derivative Test

Suppose that $f''(x)$ is continuous at c

- If $f'(x) = 0$ and $f''(x) > 0$, then $f(x)$ has a local minimum at c.
- If $f'(x) = 0$ and $f''(x) < 0$, then $f(x)$ has a local maximum at c.

Example

1. **(video)** Given $f(x) = -4x^3 + 8x^2 + 60x$, find $f'(4)$ and tell whether the function is increasing or decreasing at $x = 4$.

2. **(video)** Given $f(x) = -4x^3 + 8x^2 + 60x$
   a. Find the critical points and the intervals on increase and decrease.
   b. State whether each critical point is a maximum or a minimum.

3. **(video)** Given $f(x) = -4x^3 + 8x^2 + 60x$, find the inflection number.

4. $f(x) = 2x^3 - 3x^2 - 12x$

a. Find the critical points and the intervals on increase and decrease.

b. State whether each critical point is a maximum or a minimum.

c. Find the inflection points and the intervals on concavity.

d. Sketch the graph and verify your results.

5. $g(x) = x + 2\cos x$ on $0 \le x \le 2\pi$

a. Find the critical points and the intervals on increase and decrease.

b. State whether each critical point is a maximum or a minimum.

c. Find the inflection points and the intervals on concavity.

d. Sketch the graph and verify your results.

6. $h(x)=\dfrac{e^x}{e^x+8}$

a. Find the critical points and the intervals on increase and decrease.

b. State whether each critical point is a maximum or a minimum.

c. Find the inflection points and the intervals on concavity.

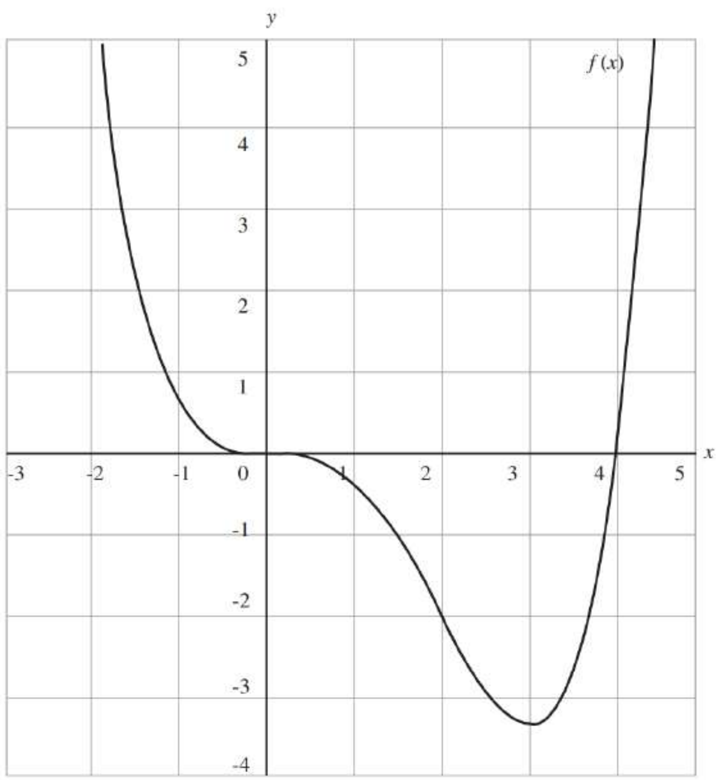
d. Sketch the graph and verify your results.

7. Suppose that $f''(x)$ is continuous on $(-\infty,\infty)$.

a. If $f'(5)=0$ and $f''(5)=6$, then f has a local_______________ at $x=5$.

b. If $f'(19)=0$ and $f''(19)=-6$, then f has a local_______________ at $x=19$.

8. Given the graph of $f(x)$, determine whether the following conditions are true.



a. $f'(3)=0$

b. $f''(x)>0$ *if* $0<x<2$

c. $f''(x)>0$ *if* $x<0$

d. $f'(x)\le 0$ *if* $x<3$

e. $f'(0)=0$

9. Find a cubic function $f(x) = ax^3 + cx^2 + d$ that has a local maximum value of 8 at $x = -2$ and a local minimum value of 6 at $x = 0$.

Curve Sketching

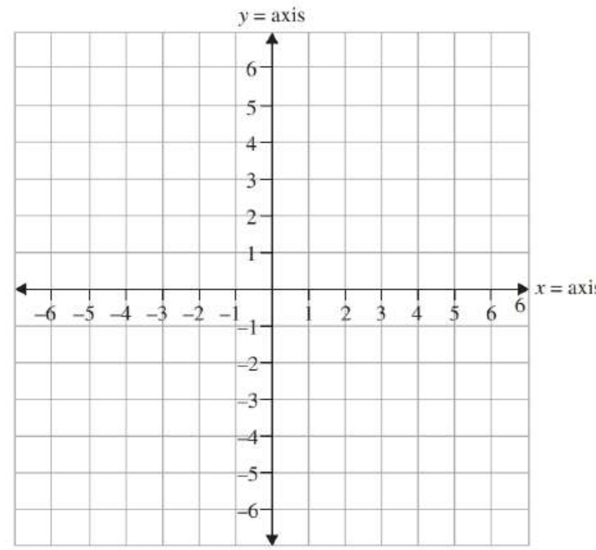
Section 4.4

A. Guidelines for sketching a curve

1. Domain
2. Intercepts (x-intercepts and y-intercepts)
3. Symmetry (Odd, even, or periodic functions)
4. Asymptotes
5. Intervals of Increase and Decrease
6. Maximum and Minimum Values
7. Intervals of Concavity

Examples

1. **(video)** $f(x)=\dfrac{x^2}{x^2-4}$. Sketch the curve using the guidelines 1 to 7.



1. Domain

2. Intercepts (x-intercepts and y-intercepts)

3. Symmetry (Odd, even, or periodic functions)

4. Asymptotes

5. Intervals of Increase and Decrease

6. Maximum and Minimum Values

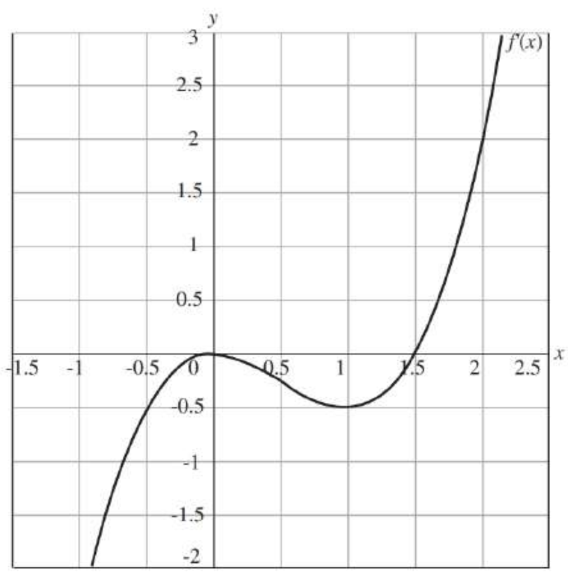
7. Intervals of Concavity

B. Guidelines for sketching a function given a sketch of its derivative

1. Find all intervals where the function is increasing and decreasing.
2. Find all intervals where the function is concave up and concave down.
3. Sketch a function that has these characteristics (there are many graphs possible).

$f(x)$ Maximum and Minimums	$\leftrightarrow$	$f'(x)$ roots
$f(x)$ Increasing	$\leftrightarrow$	$f'(x) > 0$
$f(x)$ Decreasing	$\leftrightarrow$	$f'(x) < 0$
$f(x)$ Inflection points	$\leftrightarrow$	$f'(x)$ Maximum and Minimums
$f(x)$ Concave up	$\leftrightarrow$	$f'(x)$ increasing
$f(x)$ Concave down	$\leftrightarrow$	$f'(x)$ decreasing

2. Given the graph of $f'(x)$, determine whether the following conditions are true



a. f is concave downward on the interval $(-\infty,0)$

b. f has a local maximum at $x=0$

c. f is decreasing on the interval $(-\infty,1.5)$

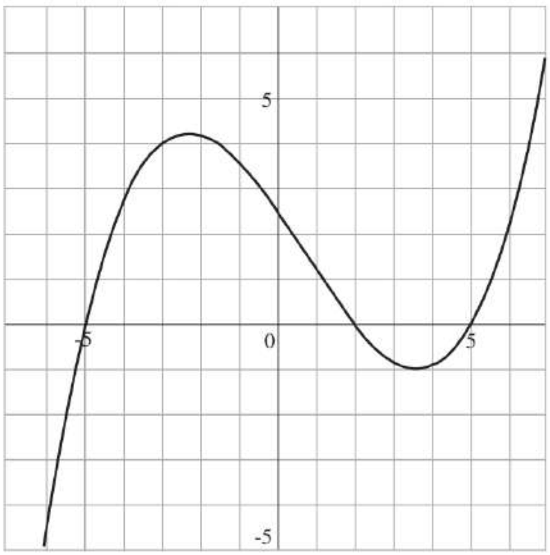
d. f has an inflection point at $x=0$

e. f is decreasing on the interval $(0,1)$

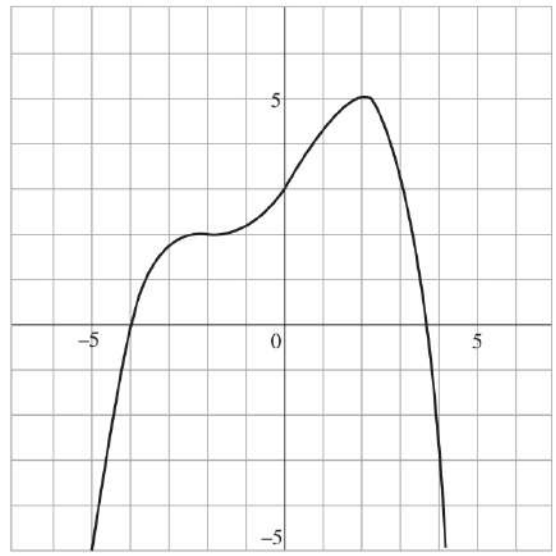
f. f is increasing on the interval $(-\infty,1.5)$

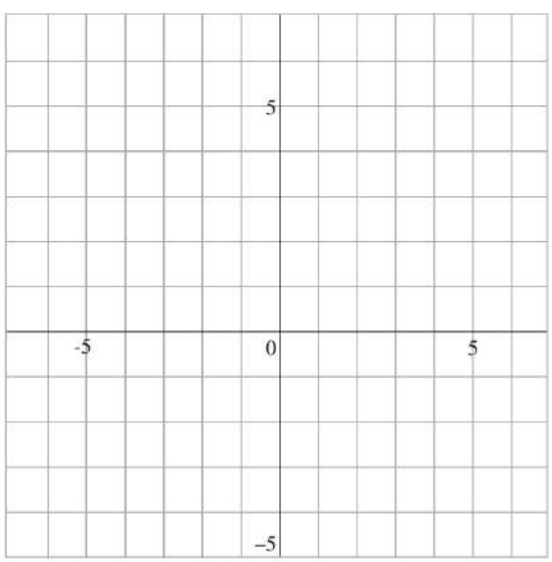
3. For each of the following is the graph of the derivative $f'(x)$ of a function. State where the original function $f(x)$ is increasing, decreasing, has max and mins, concave up and down, inflection points and then sketch $f(x)$.

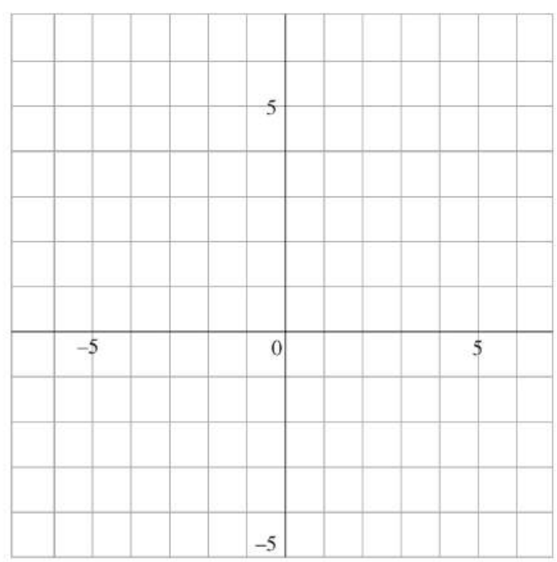
a.



b.







Optimization

Section
4.5

Steps for solving optimization problems

1. Information will be given in the problem to construct a ***constraint equation***. Usually this equation can easily be found by looking for the quantity (the actual number) given somewhere in the information. In most cases, there will be two or more variables in this equation, allowing you to solve for one of these.
2. Determine what quantity needs to be optimized (maximized or minimized). Write an equation for this quantity—this is your ***objective equation***. Your equation will again be one of at least two different variables.
3. Substitute the *constraint equation* into the *objective equation*. This will yield an objective equation of only one variable.
4. Once you have a single variable objective equation, it can be optimized by taking the derivative, setting it equal to zero and then solving for the variable.
5. Remember to answer the original question completely. You may need to plug the quantity into the objective equation to calculate a different quantity.

Hint: Make sure that units are kept constant—it is easier to equalize units at the beginning of a problem than during the procedure.

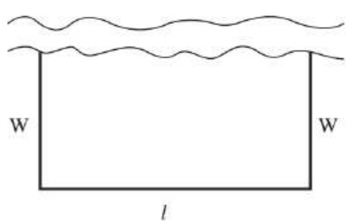
Refer to the page with geometrical formulas that was provided in Section 2.7. You are responsible for all of these formulas—they may not be provided to you during a test/exam.

Examples

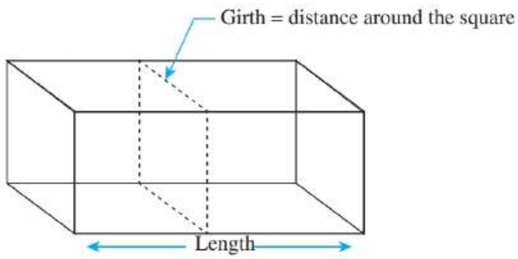
1. **(Video)** Given $f(x)=\dfrac{16x}{x^2+16}$, What is the maximum value of the function?

2. **(Video)** Find two numbers A and B (with $A \le B$) whose sum is 56 and whose product is maximized.

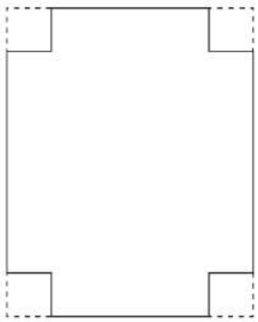
3. **(Video)** Farmer Brown has 6,000 ft of fence to create a rectangular pen that will be adjacent to a river. If he does not need to put any fence on the side that borders the river, what dimensions will maximize the area of the pen, and what is the maximum area?



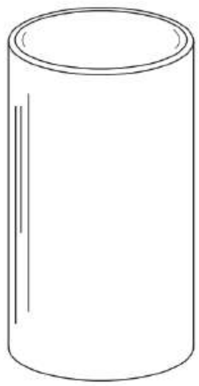
4. The U.S. Postal Service will accept a box for domestic shipping only if the sum of the length and the girth (distance around) does not exceed 108 inches. Find the dimensions of the largest volume box with a square end that can be sent.



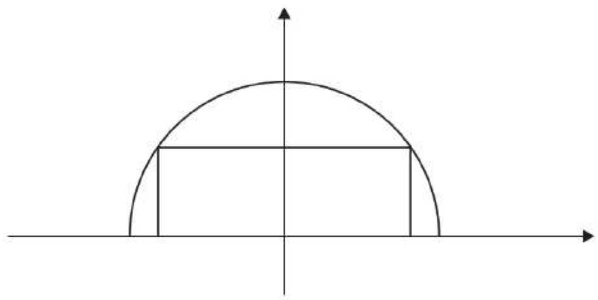
5. A box is to be made out of a 10 by 18 piece of cardboard. Squares of equal size will be cut out of each corner, and then the ends and sides will be folded up to form a box with an open top. Find the length L, width W, and height H of the resulting box that maximizes the volume (assume that $W \leq L$).



6. A cylindrical oatmeal container has a capacity of 3 L. Find the dimensions that will minimize the cost of production material to construct the container.

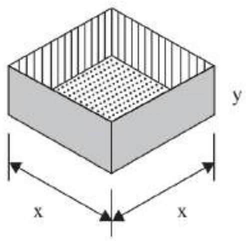


7. Find the area of the largest rectangle that can be inscribed in a semicircle with a radius 4.



8. Find the point on the line $y = 4x + 7$, which is closest to the point $(0,0)$.

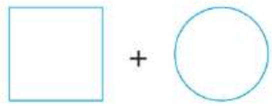
9. If 2,000 square centimeters of material is available to make a box with a square base and an open top, find the largest possible volume of the box.



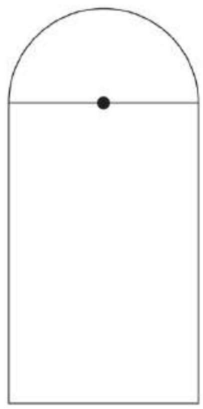
10. A piece of wire 12 m long is cut into two pieces. One piece is bent into the shape of a circle of radius r and the other is bent into a square of side s. How should the wire be cut so that the total area enclosed is:

a. Maximized

b. Minimized



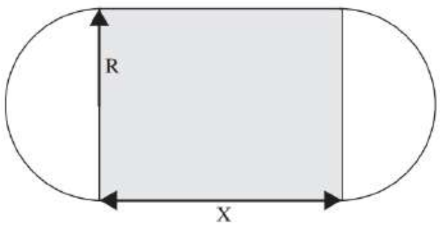
11. A Norman window has the shape of a semicircle atop a rectangle so that the diameter of the semicircle is equal to the width of the rectangle. What is the area of the largest possible Norman window with a perimeter of 45 ft?



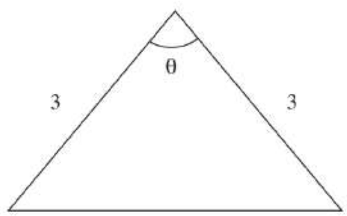
12. A running track has the shape of a rectangle with a semicircle on each end. If the length of the track is 400 m, find the dimensions so that

a. The rectangular (shaded) region is maximized

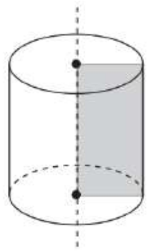
b. The entire region is maximized



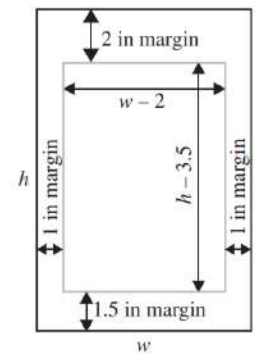
13. What angle θ between two edges of length 3 will result in an isosceles triangle with the largest area? (See diagram.)



14. Consider a rectangle of perimeter 12 in. Form a cylinder by revolving this rectangle about one of its edges. What dimensions of the rectangle will result in a cylinder of maximum volume?



15. A printer need to make a poster that will have a total area of 200 in.^{2} and will have 1 in. margins on the sides, a 2-in. margin on the top and a 1.5–in. margin on the bottom. What dimensions will give the largest printed area?



16. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees?

Newton's Method

Section 4.6

A. The Newton's Method Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Examples

1. **(Video)** Starting with $x_0 = 2$ find the third approximation x_3 to the root of the equation $x^3 + 5x - 12 = 0$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				

$x_3 =$

2. **(Video)** Starting with $x_0 = 0$ find the second and third approximation to the root of the equation $e^{-x} = 5 + 7x$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				

$x_3 =$

3. Starting with $x_0 = 1$ find the third approximation x_3 to the root of the equation $\tan^{-1}(x) = 1 - x$. Round (ONLY) your final answer correct to eight decimal places.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				

$x_3 =$

4. a. Find the equation $f(x)$ that results in a solution of $\sqrt[4]{9}$

b. Find the second, third, and fourth approximations of the root to this function if $x_0 = 2$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				
3				

$x_4 =$

Antiderivatives

Section 4.7

A function $F(x)$ is called the antiderivative of $f(x)$ if $F'(x) = f(x)$

Basic rules of antidifferentiation

In general: Reverse basic rules of differentiation.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \text{ when n} \neq -1$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

*Important: *Always use proper notation!*

**Don't forget +C*

Examples: Find the antiderivative for each of the following:

1. **(Video)** $f(x) = 5x^3 - 15x^2 + 14x - 7$

2. **(Video)** $g(x) = 20x^{\frac{1}{2}} - 16x^{\frac{3}{4}}$

3. **(Video)** $h(x) = \sqrt[5]{x^2} + \dfrac{12}{x^5} + \dfrac{1}{x}$

4. **(Video)** $k(x) = \dfrac{2x^4 + 6\sqrt{x}}{x^2}$

5. **(Video)** $m(x) = 10\tan x + 6\cos x$

6. **(Video)** Find the general antiderivative F(x), if $f(x)=\dfrac{7}{1+x^2}$

7. **(Video)** Find the antiderivative of $f(x)=2x^7-4x^3$ and $F(0)=9$

8. $f(x)=2x^3-7x^2+7x-7$

9. $g(x)=6x^{\frac{1}{5}}-18x^{\frac{4}{5}}$

10. $h(x)=\sqrt[7]{x^2}+\dfrac{2}{x^5}+\dfrac{1}{x}$

11. $k(x)=\dfrac{x^4+8\sqrt{x}}{x^2}$

12. $m(x)=9\sin x-8\cos x+\sec^2 x$

13. $k(x) = -\dfrac{2}{\sqrt{1-x^2}}$

14. Find the function $F(x)$ given that $f(x) = 2x^7 - 4x^3$ and $F(0) = 19$.

15. Find the function $f(x)$ given that $f''(x) = 24x^2 + 10$, $f(0) = 5$ and $f'(1) = 2$.

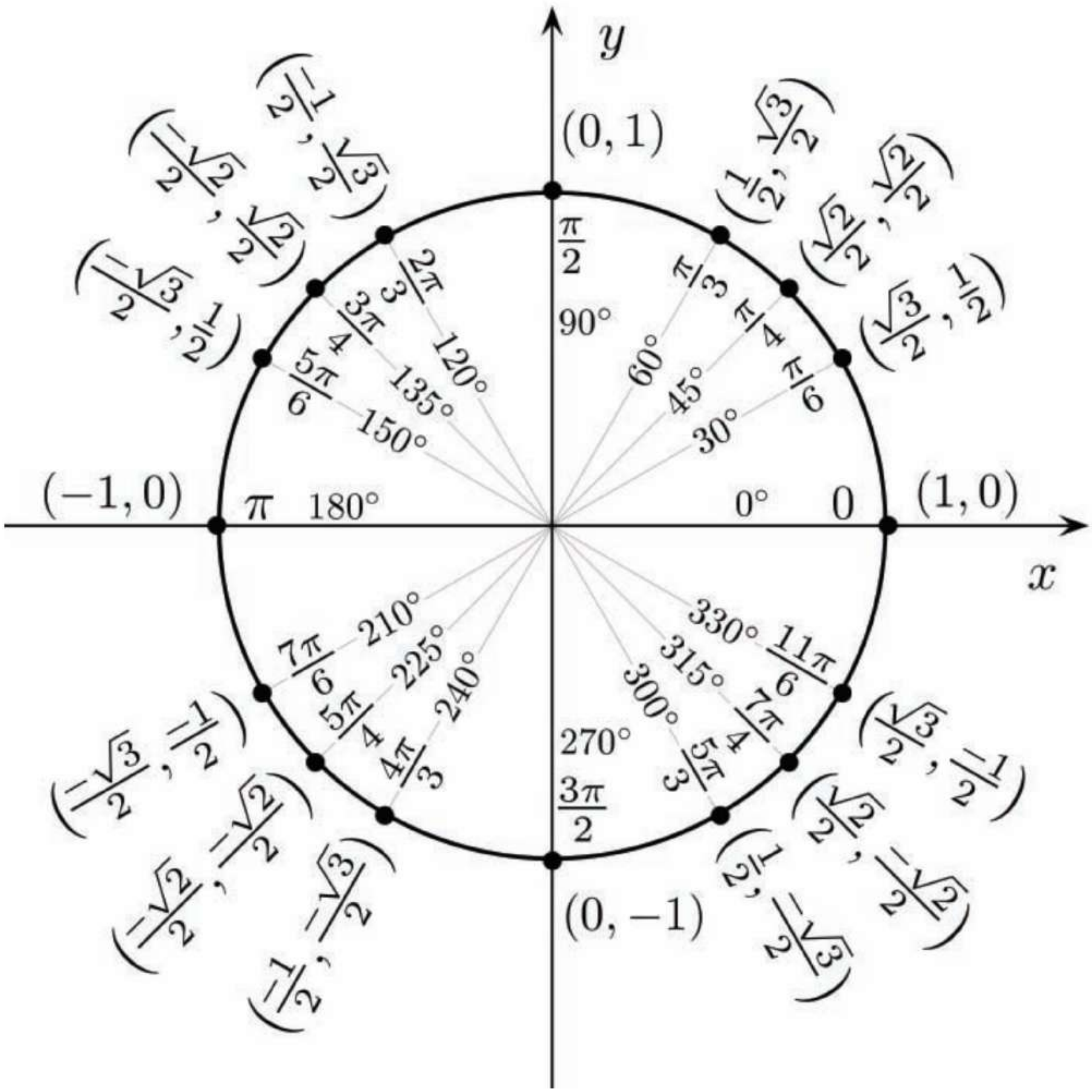
16. A particle is moving with acceleration $a(t) = 12t + 2$ measured in m/s^2. Its position at time $t = 0$ is $s(0) = 11$ and its velocity at time $t = 0$ is $v(0) = 9$. What is its position at time $t = 6$?

17. A stone is thrown straight down from the edge of a roof, 725 ft above the ground, at a speed of 5 ft/sec.

a. Given that the acceleration due to gravity is -32 ft/sec^2, how high is the stone 2 seconds later?

b. At what time does the stone hit the ground?

c. What is the velocity of the stone when it hits the ground?


y
x
(0, 1)
(1, 0)
(−1, 0)
(0, −1)
0
0°
π
180°
π/2
90°
3π/2
270°
π/6
30°
π/4
45°
π/3
60°
2π/3
120°
3π/4
135°
5π/6
150°
7π/6
210°
5π/4
225°
4π/3
240°
5π/3
300°
7π/4
315°
11π/6
330°
(√3/2, 1/2)
(√2/2, √2/2)
(1/2, √3/2)
(−1/2, √3/2)
(−√2/2, √2/2)
(−√3/2, 1/2)
(−√3/2, −1/2)
(−√2/2, −√2/2)
(−1/2, −√3/2)
(1/2, −√3/2)
(√2/2, −√2/2)
(√3/2, −1/2)

Trigonometric Identities

Trigonometric Functions

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y} = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x} = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{x}{y} = \frac{1}{\tan\theta}$$

Sum and Difference Formulas

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

Double Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half Angle Formulas

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \qquad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Sum and Difference Formulas

$$\sin a + \sin b = 2\sin\left(\tfrac{a+b}{2}\right)\cos\left(\tfrac{a-b}{2}\right)$$

$$\sin a - \sin b = 2\cos\left(\tfrac{a+b}{2}\right)\sin\left(\tfrac{a-b}{2}\right)$$

$$\cos a + \cos b = 2\cos\left(\tfrac{a+b}{2}\right)\cos\left(\tfrac{a-b}{2}\right)$$

$$\cos a - \cos b = -2\sin\left(\tfrac{a+b}{2}\right)\sin\left(\tfrac{a-b}{2}\right)$$

Product Formulas

$$\sin a \cos b = \tfrac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$$\cos a \sin b = \tfrac{1}{2}[\sin(a+b) - \sin(a-b)]$$

$$\cos a \cos b = \tfrac{1}{2}[\cos(a+b) + \cos(a-b)]$$

$$\sin a \sin b = \tfrac{1}{2}[\cos(a-b) - \cos(a+b)]$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

where A is the angle of a scalene triangle opposite side a.

Reduction Formulas

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\sin(\theta) = -\sin(\theta - \pi) \qquad \cos(\theta) = -\cos(\theta - \pi)$$

$$\tan(-\theta) = -\tan\theta \qquad \tan(\theta) = \tan(\theta - \pi)$$

$$\mp\sin x = \cos(x \pm \tfrac{\pi}{2}) \qquad \pm\cos x = \sin(x \pm \tfrac{\pi}{2})$$

Trigonometric Values for Common Angles

Degrees	Radians	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0°	0	0	1	0	Undefined	1	Undefined
30°	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
90°	$\pi/2$	1	0	Undefined	0	Undefined	1
120°	$2\pi/3$	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
135°	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$5\pi/6$	1/2	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2
180°	π	0	-1	0	Undefined	-1	Undefined
210°	$7\pi/6$	-1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2
225°	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$4\pi/3$	$-\sqrt{3}/2$	-1/2	$\sqrt{3}$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$
270°	$3\pi/2$	-1	0	Undefined	0	Undefined	-1
300°	$5\pi/3$	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$	$-\sqrt{3}$	2	$-2\sqrt{3}/3$
315°	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$11\pi/6$	-1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2
360°	2π	0	1	0	Undefined	1	Undefined